STATE UNIVERSITY DIMITRIE CANTEMIR

Manuscript
UDC: 512.536.7+515.122.4,004.424.62

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DISTANCES ON FREE MONOIDS AND THEIR APPLICATIONS IN THEORY OF INFORMATION

111.03 MATHEMATICAL LOGIC, ALGEBRA AND NUMBER THEORY

Doctor Thesis in Mathematics

Scientific Adviser: __________ Cioban Mitrofan, academician, professor

Author: __________ Budanaev Ivan

CHIŞINĂU, 2019

UNIVERSITATEA DE STAT DIMITRIE CANTEMIR

Cu titlul de manuscris
C.Z.U.: 512.536.7+515.122.4,004.424.62

BUDANAEV IVAN

DISTANŢE PE MONOIZI LIBERI ŞI APLICAŢIILE LOR ÎN TEORIA INFORMAŢIEI

111.03 LOGICA MATEMATICĂ, ALGEBRA ŞI TEORIA NUMERELOR

Teză de doctor în ştiinţe matematice

Conducător ştiinţific: _________ Cioban Mitrofan, academician, professor univ.

Autor: _________ Budanaev Ivan

CHIŞINĂU, 2019

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ANNOTATION

of the thesis **"Distances on Free Monoids and Their Applications in Theory of Information"**, submitted by Budanaev Ivan for Ph.D. degree in Mathematics, specialty 111.03 - Mathematical Logic, Algebra and Number Theory.

The thesis was elaborated in Moldova State University "Dimitrie Cantemir", Chişinău, 2019.

Thesis structure: the thesis is written in English and consists of: introduction, four chapters, general conclusions and recommendations, 200 bibliography titles, 114 pages of main text. The obtained results were published in 20 scientific papers.

Keywords: Alexandroff space, quasivariety of topological monoids, free monoids, invariant distance, quasi-metric, Levenshtein distance, Hamming distance, Graev distance, parallel decomposition, proper similarity, weighted mean, bisector of two strings, convexity, algorithm.

Domain of research: Distances on abstract algebraic structures.

Goals and objectives: The goal of the research is to study the problem of distances on free monoids. To achieve this goal, the following objectives were defined: elaboration of an effective method for extending the quasi-metric on free monoids; development of efficient representations of information for data analysis; implementation of innovative algorithms for solving text sequences problems; describe digital topologies on the discrete line.

The scientific novelty and originality consist in obtaining new theoretical results with applications in computer science. An effective method of distance extension on free monoids was developed, which helped to introduce the concept of parallel representation of information. This has allowed the development of the concepts of efficiency and similarity of the information sequences, as well as the construction of the sets of weighted mean and bisector of strings.

The important scientific problem solved in the research is the development of methods for studying distances on free monoids, which contribute to obtaining effective methods of representing information, applicable to solving different distance problems.

The theoretical significance is determined by the obtaining of the new results regarding the establishment of the conditions of existence of the extension of the distance on free monoids. The elaborated methods have allowed to approach the problems related to information sequences from a new point of view. New algorithms of constructing strings weighted mean and bisector were proposed. It has been established that the informational segment is not convex.

The applicative value of the paper consists in the use of the obtained theoretical results in the study of symmetric topologies on the digital line, imaging processing and construction of the centroid of a set of strings.

The implementation of the scientific results. The obtained results can be used in scientific research related to data analysis, the study of the efficiency of information representation, digital image processing. They can also be used in development of an optional course for university students related to the study of distances on abstract algebraic structures.

ADNOTARE

la teza **"Distanţe pe Monoizi Liberi şi Aplicaţiile lor în Teoria Informaţiei"**, înaintată de către Budanaev Ivan pentru obţinerea titlului de doctor în ştiinţe matematice la specialitatea 111.03 - Logică Matematică, Algebră şi Teoria Numerelor.

Teza a fost elaborată la Universitatea de Stat "Dimitrie Cantemir", Chişinău, anul 2019.

Structura tezei: teza este scrisă în limba engleză şi conţine introducere, patru capitole, concluzii generale şi recomandări, 200 titluri bibliografice, 114 pagini de text de bază. Rezultatele obţinute sunt publicate în 20 lucrări ştiinţifice.

Cuvinte cheie: Spaţiul Alexandrov, cvasivarietate de monoizi topologici, monoizi liberi, distanţă invariantă, cvasimetrică, distanţa Levenshtein, distanţa Hamming, distanţa Graev, descompunere paralelă, similaritate proprie, medie ponderată, bisectoare a două stringuri, convexitate, algoritm.

Domeniul de studiu al tezei: Distanţe pe structuri algebrice abstracte.

Scopul şi obiectivele lucrării. Scopul cercetării este de a studia problema distanţelor pe monoizi liberi. Pentru atingerea acestui scop au fost definite următoarele obiective: elaborarea unei metode eficiente de extindere a cvasimetricei pe monoizi liberi; dezvoltarea reprezentărilor eficiente a informaţiei pentru analiza datelor; implementarea algoritmilor inovativi pentru rezolvarea problemelor secvenţelor de text; descrierea topologiei digitale pe linia discretă.

Noutatea şi originalitatea ştiinţifică constau în obţinerea rezultatelor noi de ordin teoretic cu aplicaţii în informatică. A fost elaborată o metodă efectivă de extindere a distanţelor pe monoizi liberi, graţie căreia a fost introdus conceptul de descompunere paralelă a informaţiei. Această a permis dezvoltarea conceptelor de eficienţă şi similaritate ale secvenţelor informaţionale, la fel şi construcţia mulţimelor de medii ponderate şi bisectoare a stringurilor.

Problema ştiinţifică importantă soluţioantă constă în elaborarea metodelor de studiere a distanţelor pe monoizi liberi, care contribuie la obţinerea metodelor efective de reprezentare a informaţiei, aplicabile la soluţionarea diferitor probleme referitor la distanţe.

Semnificaţia teoretică este determinată de obţinerea rezultatelor noi ce ţin de stabilirea condiţiilor de existenţă a extinderii distanţei pe monoizi liberi. Metodele elaborate au permis abordarea problemelor legate de secvenţe de informaţie dintr-un nou punct de vedere. Au fost propuşi algoritmi de construcţie a mediilor ponderate şi bisectoarei a perechilor de stringuri. S-a stabilit că segmentul informaţional nu este convex.

Valoarea aplicativă a tezei constă in utilizarea rezultatelor teoretice obţinute la studiul topologiilor simetrice pe dreapta digitală, procesarea imaginelor şi construcţia centrului de greutate a mulţimei de stringuri.

Implementarea rezultatelor ştiinţifice. Rezultatele obţinute pot fi utilizate in cercetări ştiinţifice ce ţin de analiza datelor, studierea eficienţei reprezentării a informaţiei, procesarea digitală a imaginelor. De asemenea, ele pot servi drept suport pentru cursuri universitare opţionale.

АННОТАЦИЯ

диссертации **“Расстояния на свободных моноидах и их приложения в теории информации”**, представленной Иваном Буданаевым на соискание учёной степени доктора математических наук по специальности 111.03 – Математическая Логика, Алгебра и Теория Чисел. Диссертация выполнена в Государственном Университете “Димитрие Кантемир”, Кишинёв, 2019 год.

Структура работы: Диссертация написана на английском языке и содержит введение, четыре главы, заключение с рекомендациями, 200 библиографических названия, 114 страниц оцновного текста. Полученные результаты были опубликованы в 20 научных работах.

Ключевые слова: Пространство Александрова, свободные моноиды, инвариантное расстояние, квазиметрика, расстояния Левенштейна, Хэмминга и Граева, параллельное разложение, надлежащее сходство, взвешенное среднее, биссектриса двух строк, выпуклость, алгоритм.

Область исследования: Расстояния на абстрактных алгебраических структурах.

Цель исследования является изучение проблемы расстояний на свободных моноидах, для достижение которого определены следующие задачи: разработка эффективного метода продолжения квазиметрики на свободные моноиды; разработка эффективных представлений информации для анализа данных; внедрение инновационных алгоритмов для решения задач текстовых последовательностей; описание цифровых топологии на дискретной прямой.

Научная новизна и оригинальность заключаются в получении новых теоретических результатов с приложениями в информатике. Разработан эффективный метод продолжения расстояний на свободных моноидах, который позволил ввести концепцию параллельного представления информации, эффективности и сопоставимости информационных последовательностей, а также построить множества взвешенного среднего и биссектрисы строк.

Важной научной задачей, решаемой в исследовании, является разработка методов исследования расстояний на свободных моноидах, которые способствуют получению эффективных методов представления информации, применимых для решения задач с расстояниями.

Теоретическая значимость определяется получением новых результатов, касающихся установления условий существования продолжения расстояний на свободных моноидах. Разработанные методы позволили подойти к проблемам, связанным с информационными последовательностями, с новой точки зрения. Предложены новые алгоритмы построения взвешенного среднего и биссектрисы строк. Установлено, что информационный сегмент не является выпуклым.

Прикладная ценность работы заключается в использовании полученных теоретических результатов при исследовании симметричных топологий на цифровой прямой, обработке изображений и построении центроида множества строк.

Реализация научных результатов. Полученные результаты могут быть использованы в научных исследованиях, связанных с анализом данных, изучением эффективности представления информации, цифровой обработкой изображений. Они также могут быть использованы при разработке факультативного курса для студентов университетов, связанного с изучением расстояний на абстрактных алгебраических структурах.

INTRODUCTION

This thesis presents theoretical results of the study of distances on abstract algebraic structures. The applicative part of the research can be used in information theory, where it is necessary to define the measure similarity between data and the efficiency of data representation. These notions, in their turn, can be obtained by applying distance between the information sequences.

Hamming, Graev and Levenshtein research work bring us to the need to develop methods of extension of distances on the alphabet A over the free monoid $L(A)$. It is important for the extension to be invariant. These facts determine **the actuality and importance of the research topic.**

Different types of distances were examined by M. Frechet, V. Niemytzcki, P.S. Alexandroff, R.W. Heath, A.V. Arhangelskii, M.M. Choban, P.S. Kenderov, S. Nedev, W.A. Wilson (see [93, 152, 153, 6, 8, 9, 109, 14, 52, 15, 16, 69, 147, 199]). In the class of distances, quasi-metrics are highlighted by the fact that they are not symmetric but satisfy the condition of the axiom of the triangle inequality. It is important that any T_0-topology can be described using some quasi-metric. Discrete quasi-metrics bring us to the concept of digital space and more general to Alexandroff space. Description of abstract information systems use ordered sets. The general theory of these systems was conceived by D. Scott and Yu. Ershov (see [88, 90, 89, 91, 170, 171, 172, 174, 175, 173]).

The study of information systems is related to language theory and monoids theory. Any information represented in a given alphabet A is a sequence of elements in A. We also admit the neutral element that determines the empty symbol. This allows the representation of same sequence of information in multiple ways, with different lengths of the representation, which represent a special interest in analysis of similarity and distance between them. The results of this analysis lead to the definition of new concepts like parallel decompositions, semiparallel decompositions, proper similarity and efficiency of representation.

The research goals and objectives. The goal of the scientific research is to study the problem of distances on free monoids. To achieve this goal, the following objectives were defined:

- elaboration of an effective method for extending the quasi-metric on free monoids;
- development of efficient representations of information for data analysis;
- implementation of innovative algorithms for solving text sequences problems;
- describe image processing from the topological point of view;
- describe digital topologies on the discrete line.

The study of the research is conducted within the area of the algebraic and topological theories, and the methodology applied is based on the application of methods of monoids theory, distance spaces, language theory, algorithms theory and the informational systems theory.

The thesis scientific novelty and originality consist in its new theoretical results which are published in peer-reviewed scientific journals. Research results comprise of effective methods of distance extension on free monoids, which lead to the possibility of introducing the concept of parallel decomposition of strings. This has allowed the development of the concepts of efficiency and similarity of the information sequences, as well as the construction of the sets of weighted mean and bisector of strings. The degree of the novelty and originality is represented by:

- method of quasi-metric extension on free monoid $F^a(X, \mathcal{V})$;
- study of the digital and Alexandroff spaces;
- presented solutions for Maltsev problems;
- established relations between Hamming, Graev and Levenshtein distances;
- introduction of the concept of efficiency of representation;
- introduction of the concept of the optimal parallel decompositions;
- implementation of the algorithms for weighted mean and bisector construction for pairs of strings;
- proof of the non-convexity of the informational segment;
- introduction of the notion of the symmetric topology on the digital line;
- proof of the uniqueness of Khalimsky topology as minimal digital topology;
- elaboration of the digital image processing algorithm from the topological perspective, applicable in the digital space.

The important scientific problem solved in the research is the development of methods for studying distances extension over free monoids, which contribute to obtaining effective methods of information representation, applicable to solving different distance problems such as sequence alignment, proper similarity of a pair of strings, construction of weighted means and bisectors of strings.

The theoretical significance is determined by obtaining new results regarding the establishment of the conditions of existence of the extension of distances on free monoids. The elaborated methods have allowed to approach the problems related to information sequences from a new point of view. Additionally, the theoretical results permit the study of the digital line, and the minimality property of Khalimsky topology.

The applicative value of the paper consists in the use of the obtained theoretical results in the study of symmetric topologies on the digital line, imaging processing and construction of the centroid of a set of strings. Presented methods build a larger set of elements, using the method of optimal parallel decompositons.

Approval of scientific results. The scientific results obtained by the author in this thesis, were presented at national and international scientific conferences, and were published in peer-reviewed journals:

a) Articles presented at international scientific conferences:

- *Scattered and Digital Topologies in Information Sciences*. Plenary talk at the Conference of the Romanian Society of Applied and Industrial Mathematics ROMAI, CAIM 2018, Chisinau, Moldova, 20-23 September 2018 [65];
- *Scattered and Digital Topologies in Image Processing*. Conference on Mathematical Foundations of Informatics, MFOI 2018, Chisinau, Moldova, 2-6 July 2018 [61];
- *About Non-Convexity of the Weighted Mean of a Pair of Strings*. International Conference
  "Contemporary Trends in Science Development: Visions of Young Researchers", Academy of Sciences of Moldova, Chisinau, Moldova, 15 June 2018 [40];
- *On the Midset of Pairs of Strings*. International Conference on Mathematics, Informatics and Information Technologies, MITI 2018, Balti, Moldova, 19-21 April 2018 [41];
- *Measures of Similarity on Monoids of Strings*. Conference on Mathematical Foundations of Informatics, MFOI 2017, Chisinau, Moldova, 9-11 Nov 2017 [59];
- *Parallel Decompositions of Pairs of Strings and Their Applications.* Conference on Applied and Industrial Mathematics, Iasi, Romania, 14-17 Sept 2017 [58];
- *On the Bisector of a Pair of Strings*. The 4th Conference of Mathematical Society of the Republic of Moldova, dedicated to the centenary of Vladimir Andrunachievici (1917-1997) CMSM4, Chisinau, Moldova, 28 June - 2 July 2017 [42];
- *Parallel Decompositions and the Weighted Mean of a Pair of Strings*. International Conference "Contemporary Trends in Science Development: Visions of Young Researchers", Academy of Sciences of Moldova, Chisinau, Moldova, 15 June 2017 [43];
- *On Hamming Type Distance Functions*. Conference on Applied and Industrial Mathematics, CAIM 2016, Craiova, Romania, 15-18 Sept 2016 [36];
- *Distances on Monoids of Strings and Their Applications*. Conference on Mathematical Foundations of Informatics, MFOI 2016, Chisinau, Moldova, 25-31 July 2016 [55];
- *On the Theory of Free Topological Monoids and its Applications*. International Conference "Mathematics & IT: Research and Education", MITRE 2016, Chisinau, Moldova,

23-26 June 2016 [57];

- *About Distance Functions of Hamming-type*. International Conference "Contemporary Trends in Science Development: Visions of Young Researchers", Academy of Sciences of Moldova, Chisinau, Moldova, 25 May 2016 [38];
- *Invariant Distances on Free Semigroups and Their Applications*. The 20th Annual Conference of the Mathematical Sciences Society of Romania, 19-22 May 2016 [44];

b) Articles published in scientific journals, including conference proceedings:

- Choban M., Budanaev I., *About the Construction of the Weighted Means of a Pair of Strings*, Romai Journal, vol.14 n.1, 2018, p. 73 – 87 [64];
- Budanaev I., *About the Construction of the Bisector of Two Strings*, Romai Journal, vol.13 n.2, 2017, p. 1 – 11 [39];
- Choban M., Budanaev I., *Efficiency and Penalty Factors on Monoids of Strings*, Computer Science Journal of Moldova, vol.26 n.2 (77), 2018, p. 99 – 114 [63];
- Choban M., Budanaev I., *Distances on Free Semigroups and Their Applications*, Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica, n.1 (86), 2018, p. 92 – 119 [62];
- Choban M., Budanaev I., *Scattered and Digital Topologies in Image Processing*, Proceedings of the Conference on Mathematical Foundations of Informatics, MFOI 2018, July 2-6, 2018, Chisinau, Republic of Moldova, Chisinau, p. 21–40, 2018, ISBN: 978–9975–4237–7–9 [61];
- Budanaev I., *About Non-Convexity of the Weighted Mean of a Pair of Strings*, International Conference "Contemporary Trends in Science Development: Visions of Young Researchers", 7th Edition, Chisinau, 15 June 2018, Proceedings vol. 1, p. 6–10 [40];
- Budanaev I., *On the Midset of Pairs of Strings*, International Conference on Mathematics, Informatics and Information Technologies Dedicated to the Illustrious Scientist Valentin Belousov, MITI 2018, Communications, p. 134–135 [41];
- Choban M., Budanaev I., *Measures of Similarity on Monoids of Strings*, Conference on Mathematical Foundations of Informatics: Proceedings MFOI 2017, November 9-11, 2017, Chisinau, Republic of Moldova, Chisinau, Institute of Mathematics and Computer Science, p. 51–58, 2017, ISBN: 978–9975–4237–6–82 [59];
- Choban M., Budanaev I. *About applications of topological structures in computer sciences and communications*, Acta Et Commentationes, Ştiinţe Exacte şi ale Naturii,

Revistă Ştiinţifică, nr.2 (4), 2017, p. 45–59. [60]

- Choban M.M., Budanaev I.A., *Parallel Decompositions of Pairs of Strings and Their Applications*, Presented in plenary at CAIM 2017: The 25^{th} Conference on Applied and Industrial Mathematics, Iaşi, Romania, September 14-17, 2017, Book of Abstracts, p. 64–65 [58];
- Budanaev I., *On the Bisector of a Pair of Strings*, Conference of Mathematical Society of Moldova: Proceedings CMSM4 2017, June 25 – July 2, 2017, Chisinau, Republic of Moldova, p. 475–478, 2017, ISBN: 978-9975-71-915-5 [42];
- Budanaev I., *Parallel Decompositions and The Weighted Mean of a Pair of Strings*, International Conference "Contemporary Trends in Science Development: Visions of Young Researchers", 6th Edition, Chisinau, 15 June 2017, Proceedings, p. 7–11 [43];
- Budanaev I., *On Hamming Type Distance Functions*, CAIM 2016: The 24^{th} Conference on Applied and Industrial Mathematics, Craiova, Romania, September 15-18, 2016, Book of Abstracts, Editura SITECH, Craiova, p. 15–16 [36];
- Budanaev I., *On Hamming Type Distance Functions*, Romai Journal v.12, no.2 (2016), p. 25 – 32 [37];
- Choban M., Budanaev I., *About Applications of Distances on Monoids of Strings*, Computer Science Journal of Moldova, vol.24 n.3 (72), 2016, p. 335 – 356 [56];
- Budanaev I., Choban M., *Invariant Distances on Free Semigroups and Their Applications*, 20th Annual Conference of the Mathematical Sciences Society of Romania, Rezumatele Comunicărilor, A XX-a Conferinţă Anuală a Societăţii de Ştiinţe Matematice din România, Baia Mare, 19-22 mai 2016, p. 17–18, [44];
- Choban M., Budanaev I., *On the Theory of Free Topological Monoids and its Applications*, International Conference "Mathematics & Information Technologies: Research and Education". MITRE 2016 Abstracts. Chisinau, June 23-26, 2016, p. 21–22 [57];
- Choban M., Budanaev I., *Distances on Monoids of Strings and Their Applications*, Proceedings of the Conference on Mathematical Foundations of Informatics, MFOI 2016, July 25-29, 2016, Chisinau, Republic of Moldova, p. 144–159, 2016, ISBN: 978–9975–4237–4–8 [55];
- Budanaev I., *About Distance Functions of Hamming Type*, International Conference "Contemporary Trends in Science Development: Visions of Young Researchers", Conferinta Stiintifica a Doctoranzilor, IMI ASM, 2016, p. 296 – 301 [38];

c) Awards and scholarships for the results of the scientific research:

- *Best Presentation Award* for the Communication "On the Midset of Pairs of Strings", International Conference on Mathematics, Informatics and Information Technologies, MITI 2018, Balti, Moldova.
- *World Federation of Scientists Scholarship* to conduct research on the topic “Distances on Abstract Algebraic Systems and their Applications“, related to the WFS Planetary Emergency “Science and Technology“, 2018.
- *Academic Excellence Scholarship* offered by the Government of Republic of Moldova, 2017.
- *1st Place Award* on International Conference “Contemporary Trends in Science Development“ for article “About Distance Functions of Hamming Type“, June 2017.
- *The Young Researcher Prize for the Best Paper* for the article “Distances on Monoids of Strings and Their Applications“ on Conference on Mathematical Foundations of Informatics MFOI 2016.

A total of 20 scientific works were published, comprising 7 articles in peer-reviewed scientific journals (2 articles with no co-authors) and 13 conference theses.

Summary of the thesis chapters. Thesis structure is represented by four chapters containing theoretical results on methods of distance extension over free monoids, algorithms for constructing the weighted mean and bisector sets of pairs of strings, study of the problem of informational segment convexity and digital image analysis with a topological approach. Additionally, thesis contains annotations in English and Romanian, introduction, general conclusions and recommendations, bibliography list with 200 titles.

In the introduction, the actuality and importance of the research topic are formulated. In addition, the research goals, objectives, the scientific novelty and originality are stated. The scientific problem under study is presented with the emphasis on the importance of the theoretical and applicative value of the work. A brief analysis of the problems and publications on the thesis topic is given. This sections concludes with a summary of the content of the paper.

The first chapter of the thesis has an introductory character and aims at presenting the current situation in the field of distance spaces. It defines and classifies distances, distance spaces, informational systems of Scott-Ershov type, universal topological algebras, spaces of strings. Additionally, in this chapter, Maltsev problems are formulated for free monoids. At the end of the chapter are formulated the scientific research problem, various particular cases that are the object of study in subsequent chapters. The research problem is formulated, the purpose and objectives of the research are established.

In second chapter, quasivarieties of topological monoids are studied. The definition of the non-Burnside quasivariety is given. It is established that for any non-Burnside quasivariety $\mathcal{V}$ and any quasi-metric ρ on a set X with basepoint p_X on free monoid $F^a(X, \mathcal{V})$ there exists a unique stable quasi-metric $\hat{\rho}$ with the properties:

(a) $\rho(x, y) = \hat{\rho}(x, y)$ for all $x, y \in X$.

(b) If d is an invariant quasi-metric on $F^a(X, \mathcal{V})$ and $d(x, y) \leq \rho(x, y)$ for all $x, y \in X$, then $d(x, y) \leq \hat{\rho}(x, y)$ for all $x, y \in F^a(X, \mathcal{V})$.

(c) If ρ is a metric, then $\hat{\rho}$ is a metric as well.

(d) If $Y \subseteq X$, $d = \rho|Y$ and $\hat{d}$ is the maximal invariant extension of d on $F^a(Y, \mathcal{V})$, then $F^a(Y, \mathcal{V}) \subseteq F^a(X, \mathcal{V})$ and $\hat{d} = \hat{\rho}|F^a(Y, \mathcal{V})$.

(e) For any quasi-metric ρ on X and any points $a, b \in F^a(X, \mathcal{V})$ there exists $n \in N$ and representations $a = a_1 a_2 ... a_n$, $b = b_1 b_2 ... b_n$ such that $a_1, b_1, a_2, b_2, ..., a_n, b_n \in X$ and $\hat{\rho}(a, b) = \sum\{\rho(a_i, b_i) : i \leq n\}$.

This theorem is applied to solving Maltsev problems for free monoids. It is important to mention that the Maltsev's embedding problem is solved for T_0-topologies. Additionally, it is proved that the space X is Alexandroff or digital if and only if $F^a(X, \mathcal{V})$ is an Alexandroff or digital space.

In chapter 3 it is proved that there are invariant distances on the monoid $L(A)$ of all strings closely related to Levenshtein distance. A distinct definition of the distance on $L(A)$ is introduced, based on the Markov-Graev method, proposed by him for free groups. In result, it is shown that for any quasi-metric d on alphabet A in union with the empty string there exists a maximal invariant extension d^* on the free monoid $L(A)$. This new approach allows to introduce parallel and semiparallel decompositions of two strings. In virtue of Theorem 3.3.1, they offer various applications of distances on monoids of strings in solving problems from distinct scientific fields. As an example, the study of the measure of proper similarity is approached from a new perspective. The notions of efficiency and penalty of strings alignment are introduced.

Chapter 4 describes geometrical and topological aspects of information analysis based on the results of the previous chapter. The problem of the weighted means between two strings is formulated and the algorithm of construction is presented. As an intuitive consequence, the question of the convexity of the set of the weighted means is studied for Hamming and Graev distances. As an additional application of the optimal parallel string decompositions, the algorithm for constructing the bisector of a pair of strings is presented. The chapter concludes with a digital image processing algorithm developed from a topological point of view using the notion of scattered topologies.

The **General conclusions and recommendations** section outlines the general conclusions of the author on the results obtained within the thesis. Conclusions are presented in form of obtained results, and are followed by author's recommendations on how these results can be applied to various scientific domains, as well as prospective research.

1. CURRENT SITUATION IN THE FIELD OF QUASI-METRIC SPACE THEORY AND THEIR APPLICATIONS IN ALGEBRA AND INFORMATION THEORY

In the XXI century, the value of information and the need of sending and storing it in an intact and secret form increased. In solving these issues mathematics and computer science prove to be useful. New software applications are developed on a regular basis on top of data analysis algorithms and libraries. These ensure full user anonymity and information security on both servers and mobile phones available for each citizen. Financial transactions, ATM operations, digital currencies - all are based on such systems. Also, the applications of such algorithms can be found in our daily lives not related to finance. Under the hood of even a basic text editor one can find implemented different information analysis algorithms. Various messengers, electronic mail services, short message services, data transfer protocol - would not provide guarantee of information integrity and delivery without most popular distance function taught in high school. Every instance we search something on the internet, the services providers use complex algorithms that analyze and categorize information.

One of the needs that rise in contemporary world is the development of algebraic methods, such as numerical analysis and information encoding.

Furthermore, the dynamic transition of our technological civilization to digital processing and data transmission systems created many problems in the design of modern systems in computer science and telecommunications. Providing robustness and noise immunity is one of the most important and difficult tasks in the data transmission, recording, playback and storage. The distance between information plays nowadays a paramount role in mathematics, computer science and other interdisciplinary areas. The first among many scientists in the field, who presented the theoretical solutions to error detecting and error correction problems, were C. Shannon, R. Hamming and V. Levenshtein (see [177, 106, 130]). The modern problems of informatics are transfer, storage, protection and processing of information.

One of the modern domain of the information processing use extensively text algorithmics. Algorithms for text (strings) have long been studied in computer science, and analysis of the data from molecular sequences underlies bioinformatics. Existing and emerging string manipulation algorithms provide a significant crossing of informatics and molecular biology. This field covers a wide range of string algorithms from classical computer science to modern molecular biology and, if possible, integrates these two fields. Biology problems that arise in real-life world are transformed into strings and solved with help of different methods. The transaction from biology to computer science and its applications that arise, however, is for some people intuitive, while for others a miracle. Our daily life depends on information technology, on information coding, and packets transmission. Biology and genetics are not an exception:

"The digital information that underlies biochemistry, cell biology, and development can be represented by a simple string of G's, A's, T's and C's. This string is the root data structure of an organism's biology" .[154]

"In a very real sense, molecular biology is all about sequences. First, it tries to reduce complex biochemical phenomena to interactions between defined sequences . . . ". [110]

Interdisciplinary science, where computer science and mathematics find their applications in other domains, are contemporary fields where much effort is required from scientists to achieve success. The tasks that arise are to study relevant theory, protocols and existing algorithmic methods already used, as well as to look for the ideas that aren't already used, but which can be successfully applied in solving current problems. One good example of such interdisciplinary field that applies mathematics and computer science models in solving domain problems is phylogeny, where many algorithms of string matching and pattern analysis find use. A good collection of such algorithms one can find in Dan Gusfields' work [102, 103, 104].

Another interdisciplinary application are images processing systems, which prove to be useful when applied in computer-aided diagnosis schemes based on artificial intelligence techniques and methods applied in the diagnosis process by S. Cojocaru, C. Gaindric, I. Titchiev, L Butseva, and others [46, 75, 95].
In connection with current exponential growth of information volume these problems remain on the radar. Theoretical computer science is based on different areas of mathematics. In particular, algebraic structures play an important role in solving problems related to information.

The history of the data storage, verification and correction is first mentioned with the precise copying of the Jewish bible, beginning before Christ. The groups of Hebrew who worked on holy scripts between 6th and 10th century CE were called Masoretes. This group of scribe-scholars contributed to the Numerical Masorah. In classical antiquity, copyists were paid for their work according to the number of stichs (lines of verse). As the prose books of the Bible were hardly ever written in stichs, the copyists, in order to estimate the amount of work, had to count the letters. For the Masoretic Text, such statistical information more importantly also ensured accuracy in the transmission of the text with the production of subsequent copies that were done by hand.

The oldest extant manuscripts of the Masoretic Text date from approximately the 9th century CE. The Masoretes devised the vowel notation system for Hebrew that is still widely used, as well as the trope symbols used for cantillation.

To describe a more contemporary period of the field, we must mention one of the most important scientist Claude Shannon, who is also known as "the father of information theory"[119]. Shannon is known for having founded information theory with a milestone in his article "A Mathematical Theory of Communication" [177], which he published in 1948. He is perhaps equally well known for creating the theory of designing digital circuits in 1937. He wrote his thesis demonstrating that electrical applications of Boolean algebras can construct any logical and numerical relations. Shannon contributed to the field of cryptanalysis for national defense during the Second World War, including his fundamental work on coding and securing telecommunications. In addition to distance functions, algebraic abstract structures are also extensively used in cryptanalysis by V. A. Shcherbacov, P. Syrbu, and V. S. Vostokov (see [167, 168, 129, 169, 178, 179, 192, 193]).

Another important contribution in development of information theory was brought by Richard Hamming - U.S. mathematician, whose work in the field of information theory had a significant impact on computer science and telecommunications. The main contribution is the *Hamming code*, and the *Hamming distance*. In 1950, Richard Hamming publishes *Error Detecting and Error Correcting Codes* [106].

One of the Russian scientists who achieved important results in the domain was academician Vladimir Kotelnikov, who is the founder of the theory of potential noise immunity. The sampling theorem (Kotelnikov's theorem) has become known worldwide and widely used. He developed the basics and created telemetry equipment for aircraft and missiles, as well as for radio-location of the planets of the solar system. He played a major role in the development of Russian science as director of the Institute of Radio Electronics of the Russian Academy of Sciences and vice-president of the Russian Academy of Sciences. In 1956, Vladimir Kotelnikov publishes *Theory of Potential Noise Immunity* [125], which has a great merit in the formulation and development of fundamental research in such areas as noise immunity of radio systems and statistical radiophysics.

Another scientist with important results in theory of coding is Vladimir Levenshtein. His area of scientific interest included:

- Optimizational and combinational problems of coding and testing.
- Universal limits for codes and designs and the theory of orthogonal polynomials.
- Synchronization properties of sequences, codes and automata.
- Perfect codes that correct single errors of different types.

Vladimir Levenshtein has provided the best-known universal bounds to optimal sizes of codes and designs in metric spaces, including the Hamming space and the Euclidean sphere. In 1965, Vladimir Levenshtein publishes *Binary codes capable of correcting deletions, insertions, and reversals* [130], where he introduced *Levenshtein distance*. The Levenshtein distance (also called edit or editing distance) between two lines in information theory and computer linguistics is the minimum number of insertion operations of one character, the removal of one character and the replacement of one character with another, necessary for converting one line to another. It lays at the root of today's spell-checking software applications. Vladimir Levenshtein also contributed to the basic technology used in the third generation of wired cellular telephony.

The notions of space, distance and functions are important concepts that have evolved over time during the development of mathematics and information theory from antiquity to the present. Topological spaces and distances on spaces have emerged from the need to study convergence. Notions of open and closed sets, neighborhoods of points appeared in a natural manner. Functions or applications also determine the linkage between spaces and mobility across spaces. Algebraic operations on spaces are special functions and create special relationships between "elements cohorts" and elements on the same space.

In this chapter:

– are presented important notions and main results, which are discussed in depth in the following chapters;

– is included a synthesis of the scientific research presented in the literature on the evolution of the development of analysis methods of spaces with distance and study of information sequences.

1.1. Distance spaces

Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ be a mapping such that for all $x, y \in X$ we have:

(i_m) $d(x, y) \geq 0$;

(ii_m) $d(x, y) + d(y, x) = 0$ if and only if $x = y$.

Then (X, d) is called a *distance space* and d is called a *distance* on X.

General problems of the distance spaces were studied in [8, 9, 14, 29, 49, 94, 147, 148, 149, 150, 152, 153]. The function $d : X \times X \to \mathbb{R}$ with the property (i_m) is called a pseudo-distance on a set X.

The notion of a distance space is more general than the notion of o-metric spaces in sense of A. V. Arhangel'skii [14] and S. I. Nedev [147]. A distance d is an o-metric if from $d(x, y) = 0$ it follows that $x = y$. These notions coincide in the class of T_1-spaces.

Let d be a pseudo-distance on X and $B(x, d, r) = \{y \in X : d(x, y) < r\}$ be the *ball* with the center x and radius $r > 0$. The set $U \subset X$ is called *d-open* if for any $x \in U$ there exists $r > 0$ such that $B(x, d, r) \subset U$. The family $\mathcal{T}(d)$ of all d-open subsets is the topology on X generated by d. The space $(X, \mathcal{T}(d))$ is a *sequential space*, i.e. a set $B \subseteq X$ is closed if and only if together with any sequence it contains all its limits [87]. If d is a distance on X, then $(X, \mathcal{T}(d))$ is a T_0-space and vice versa.

The set $B(x, d, r) = \{y \in X : d(x, y) < r\}$ is the *closed ball* with the center x and radius $r > 0$.

Let (X, d) be a distance space, $\{x_n : n \in \mathbb{N} = \{1, 2, ...\}\}$ be a sequence in X and $x \in X$. We say that the sequence $\{x_n : n \in \mathbb{N}\}$:

1. is *convergent* to x if and only if $lim_{n\to\infty} d(x, x_n) = 0$. We denote this by $x_n \to x$ or $x = lim_{n\to\infty} x_n$ (also, we may denote by $x \in lim_{n\to\infty} x_n$);

2. is *convergent* if it converges to some point in X;

3. is *Cauchy* or *fundamental* if $lim_{n,m\to\infty} d(x_n, x_m) = 0$.

A distance space (X, d) is *complete* if every Cauchy sequence in X converges to some point in X.

Lemma 1.1.1. *Let (X, d) be a distance space, $\varphi : X \longrightarrow X$ be a mapping and for each point $x \in X$ there exist two positive numbers $c(x), k(x) > 0$ such that $d(\varphi(x), \varphi(y)) \leq k(x) \cdot d(x, y)$ provided $y \in X$ and $d(x, y) \leq c(x)$. Then the mapping φ is continuous.*

Proof. Let $\{x_n \in X : n \in \mathbb{N}\}$ be a convergent to $x \in X$ sequence. Then $lim_{n\to\infty} d(x, x_n) = 0$, $lim_{n\to\infty} d(\varphi(x), \varphi(x_n)) = 0$ and $lim_{n\to\infty} \varphi(x_n) = \varphi(x)$. Hence the mapping φ is continuous. □

Let X be a non-empty set and d be a distance on X. Then:

- (X, d) is called a *symmetric space* and d is called a *symmetric* on X if for all $x, y \in X$ we have (iii_m) $d(x, y) = d(y, x)$;

- (X, d) is called a *quasi-metric space* and d is called a *quasi-metric* on X if for all $x, y, z \in X$ we have (iv_m) $d(x, z) \leq d(x, y) + d(y, z)$;

- (X, d) is called a *metric space* and d is called a *metric* if d is a symmetric and a quasi-metric simultaneous.

The following theorem is well-known.

Theorem 1.1.1. *For each topology $\mathcal{T}$ on a set X there exists a family $\{d_\mu : \mu \in M\}$ of pseudo-quasi-metrics on X which generated the topology $\mathcal{T}$, i.e. $\mathcal{T} = sup\{\mathcal{T}(d_\mu) : \mu \in M\}$.*

Proof. Let B be an open base of the space X. For any open set U of X we consider the function $d_U : X \times X \to \mathbb{R}$, where:

- $d_U(x, y) = 1$ for $x \in U$ and $y \in X \setminus U$;

- $d_U(x, y) = 0$ for $x \in X \setminus U$;

- $d_U(x, y) = 0$ for $x, y \in U$.

The function d_U is a pseudo-quasi-metric and $T(d_U) = \{\emptyset, U, X\}$. Hence $T = sup\{T(d_U) : U \in B\}$. The proof is complete. □

Therefore, quasi-metrics and pseudo-quasi-metrics play an important role in the study of topological spaces.

Considering topology, every distance d on a non-empty set X determines some geometry on X. These facts allow to apply theory of spaces with distance in different theoretical and application fields (see [14, 92, 28, 29, 25, 36, 37, 38, 72]).

Let X be a non-empty set and d be a distance on X.

Definition 1.1.2. Let point $y \in Y$ lie between points $x, z \in X$ and denote by $(xz)y$ if $d(x, y) + d(y, z) = d(x, z)$. The set $[x, z]_d = \{y \in X : (xz)y\}$ is called a segment with endpoints in x and z.

We notice that x is the origin of the segment $[x, z]_d$, and z is the terminal point of this segment. In general, $[x, z]_d \neq [z, x]_d$, and $x, z \in [x, z]_d$.

Definition 1.1.3. The set $H \subset X$ is called d-convex, if $[x, y]_d \cap [y, x]_d \subset H$ for any two points $x, y \in H$.

Let $a, b \in \mathbb{R}$ and $a < b$. Application $f : [a, b] \longrightarrow X$ is called a "path". For any path $f : [a, b] \longrightarrow X$, its length is given by $l_d(f) = sup\{d(f(a), f(t_1)) + d(f(t_1), f(t_2)) + \ldots + d(f(t_i), f(t_{i+1})) + \ldots + d(f(t_n), f(b)) : n \in \mathbb{N}, a \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq b\}$. For path $f : [a, b] \longrightarrow X$, its inverse path is given by $f^{-1} : [a, b] \longrightarrow X$, for which $f^{-1}(t) = f(a + b - t)$ for any $t \in [a, b]$. In general, we have that $l_d(f) \neq l_d(f^{-1})$. If the distance d is symmetric, then $l_d(f) = l_d(f^{-1})$ for any path f.

The path $f : [a, b] \longrightarrow X$ joins the point $f(a)$ with the point $f(b)$. If any two points can be joined with a path, then the space is called the *path space*. The path $f : [a, b] \longrightarrow X$ is called an *arc*, if $f(p) \neq f(q)$ for any $a \leq p < q < b$ or $a < p < q \leq b$. We say that this arc joins the point $f(a)$ with the point $f(b)$. If $f(a) = f(b)$, then this arc is called the loop in point $f(a)$.

The path $f : [a, b] \longrightarrow X$ is called an *isometric arc*, if $d(f(p), f(q)) = |p - q|$ for any $p, q \in [a, b]$. On any isometric arc $f([a, b])$ distance d is a metric.

The space (X, d) is called convex if any two different points can be joined with an isometric arc. In this case d is symmetric. The convex space is the path-convex, and the path-convex space is topologically convex.

A distance space X is said to be hyperconvex if it is convex and its closed balls have the binary Helly property. That is:

1. any two points x and y can be connected by the isometric image of a line segment of length equal to the distance between the points;

2. if $F = \{\bar{B}(p_i, d, r_i) : I \in M\}$ is any family of closed balls such that each pair of balls in F meet, then there exists a point x common to all the balls in F.

Equivalently, if a set of points p_i and radii $r_i > 0$ satisfies $r_i + r_j \geq d(p_i, p_j)$ for each i and j, then there is a point q of the distance space that is within distance r_i of each p_i.

In Euclidean space any isometric arc is a segment and is d-convex. One can observe that a space with distance (X, d) can be a convex space, but not any isometric arc can be d-convex.

Example 1.1.1. *Let $X = \{(x, y) : x, y \in \mathbb{R}\}$ be the arithmetic plane with distance $d((x, y), (u, v)) = sup\{|x - u|, |y - v|\}$. Then $f : [0, 1] \longrightarrow X$, where $f(t) = (t, t)$ for any $t \in [0, 1]$ is an isometric arc with origin $(0, 0)$ and terminal point $(1, 1)$. The set $H = f([0, 1])$ is not d-convex because $[(0, 0), (1, 1)]_d = \{x, y) \in X : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.*

Distance d is called discrete, if $d(x, y) \in \{0, 1\}$ for any $x, y \in X$. Data processing in Informatics involves finite spaces with discrete distances.

Example 1.1.2. *Let X be a set, $d(x, x) = 0$ for any $x \in X$ and $d(x, y) = 1$ for any two distinct points $x, y \in X$. Then d is a discrete metric on X. We have that $[x, y]_d = \{x, y\}$ for any $x, y \in X$. Therefore, any set H in (X, d) is d-convex. The space (X, d) is not convex or hyperconvex in Helly sense.*

Example 1.1.3. *Let X be a set with at least two points. Fix on X a relation of total order. For any $x, y \in X$ denote:*

- *$d_l(x, y) = 0$ for $y \leq x$ and $d_l(x, y) = 1$ for $x < y$;*
- *$d_r(x, y) = 0$ for $x \leq y$ and $d_r(x, y) = 1$ for $y < x$;.*

Then d_l, d_r are discrete quasi-metrics on X and $d = d_l + d_r$ is a discrete metric on X.

On discrete spaces there exist discrete distances. Moreover, on T_0 -spaces there exist discrete quasi-metrics.

A retract of a distance space (X, d) is a function $f : X \longrightarrow X$ of X to a subspace of itself, such that:

1. for all $x \in X$, $f(f(x)) = f(x)$ ($f(x) = x$ for any $x \in f(X)$);
2. for all $x, y \in X$, $d(f(x), f(y)) \leq d(x, y)$ (f is a non-expansive mapping).

A retract of a space X is a subspace of X which is an image of a retraction. A distance space X is said to be *injective* if, whenever X is isometric to a subspace Z of a quasi-metric space Y with $Z \setminus X$ is discrete, that subspace Z is a retract of Y.

However, it is the Aronszajn and Panitchpakdi theorem [20] (see also [48]) that mentions of injectivity and hyperconvex space are equivalent in the class of complete metric spaces. Moreover, every injective metric space is a complete space [20].

There is an open access electronic journal related to current topic called *"Analysis and Geometry in Metric Spaces"* that publishes cutting-edge research on analytical and geometrical problems in metric spaces and applications related topics:

– Geometric inequalities in metric spaces;

– Geometric measure theory and variational problems in metric spaces;

– Analytic and geometric problems in metric measure spaces, probability spaces, and manifolds with density;

– Analytic and geometric problems in sub-riemannian manifolds, Carnot groups, and pseudo-hermitian manifolds;

– Geometric control theory;

– Curvature in metric and length spaces;

– Geometric group theory;

– Harmonic Analysis. Potential theory;

– Mass transportation problems;

– Quasiconformal and quasiregular mappings. Quasiconformal geometry;

– Differential equations associated to analytic and geometric problems in metric spaces.

Different problems of the metric space geometry are analyzed in [45, 80, 81]. We note that the theory of metric spaces has a large spectre of applications.

1.2. On discrete spaces

Alexandroff spaces were first introduced in 1937 by P. S. Alexandroff [6] (see also [12]) under the name of discrete spaces, where he provided the characterizations in terms of sets and neighbourhoods.

A space X is called an Alexandroff space if it is a T_0-space and the intersection of any family of open sets is open [6, 12].

Let $\leq$ be a linear ordering on a set X. We define two quasi-metrics d_l and d_r on X, where $d_l(x, x) = d_r(x, x)$ for any $x \in X$ and for $x \prec y$ we put $d_l(x, y) = 1$, $d_l(y, x) = 0$, $d_r(x, y) = 0$, $d_r(y, x) = 1$. In this case $d_s(x, y) = d_r(x, y) + d_l(x, y)$ is a metric. In general, a sum of quasi-metrics is also a quasi-metric, and may not be a metric.

We observe the importance of distances with natural values. We affirm that this fact is important from topological point of view as well.

Theorem 1.2.1. *On a space X there exists a quasi-metric with the natural values if and only if X is an Alexandroff space.*

Proof. Let X be an Alexandroff space. For any $x \in X$ denote by M_x the intersection of all open sets which contains x. Then M_x is the minimal open set which contains the point $x \in X$. Observe that if $x, y \in X$, $x \neq y$, and $y \in M_x$, then $M_y \subset M_x$ and $x \notin M_y$. Consider the distance $\rho(x, y)$, where $\rho(x, x) = 0$ for any $x \in X$, $\rho(x, y) = 0$ if $y \in M_x$, and $\rho(x, y) = 1$ if $y \notin M_x$. We affirm that ρ is a quasi-metric with natural values. By construction, $\rho(x, y) \in \{0, 1\}$ and ρ has natural values. Let $x, y, z \in X$. If $\rho(x, y) = \rho(y, z) = 0$, then $y \in M_x$ and $z \in M_y \subset M_x$. Hence $\rho(x, z) = 0$. In this case $\rho(x, y) + \rho(y, z) = \rho(x, z)$. If $\rho(x, y) + \rho(y, z) \geq 1$, and given that $\rho(x, z) \leq 1$ we conclude with $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$. Therefore ρ is a quasi-metric.

If d is a quasi-metric on X with natural values, then $M_x = \{y \in X : d(x, y) < 1\}$ is the minimal open set which contains the point $x \in X$. Therefore $(X, \mathcal{T}(d))$ is an Alexandroff space, and this concludes the proof of the Theorem 1.2.1.

□

General criteria of quasi-metrizability of spaces were proved in [147].

Example 1.2.1. *Let X be a linear orderable non-empty set with the linear order $\leq$. For any $x \in X$ we put $M_x = \{y \in X : y \geq x\}$. We can consider M_x as the minimal open set which contains the point x. The sets M_x form the open base of the topology $\mathcal{T}$ on X. Then $(X, \mathcal{T})$ is an Alexandroff space. If for the set X there exists $a = infimum X$, then $(X, \mathcal{T})$ is a compact Alexandroff space. The space $(X, \mathcal{T})$ is a compact space if and only if there exists $a = infimum(X)$. The set $(X, \leq)$ is well ordered if and only if any subspace Y of the pace $(X, \mathcal{T})$ is compact.*

1.3. Abstract information systems

To describe the abstract information systems, we will use the works [22, 32, 34, 107].

A (non-strict) partial order is a binary relation $\leq$ over a set P satisfying the following axioms:

1. $a \leq a$ (reflexivity: every element is related to itself).
2. if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry).
3. if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).

A set with a partial order is called a partially ordered set or briefly a poset. If $a \leq b$ and $a \neq b$, then we put $a < b$. A totally ordered set is a poset such that for any two elements x and y either $x < y$, or $x = y$, or $x > y$.

The dual order $\leq_d$ of a partially ordered set $(P, \leq)$ is the same set P with the partial order relation replaced by its inverse: $x \leq_d y$ if and only if $y \leq x$.

For $a \preceq b$, the closed interval $[a, b]$ is the set of elements x satisfying $a \preceq x \preceq b$. It contains at least the elements a and b. Using the corresponding strict relation "$\prec$", the open interval (a, b) is the set of elements x satisfying $a \prec x \prec b$. An open interval may be empty, even if $a \prec b$. The half-open intervals $[a, b)$ and $(a, b]$ are defined similarly. A poset is locally finite if every interval is finite.

Definition 1.3.1. (Extremal elements): Suppose $(P, \preceq)$ is a poset, a and b are elements of P, and S is a non-empty subset of P. Then:

- b is a maximal element of S iff b is in S and there is no $x \in S$ such that $b \prec x$;
- b is a maximum of S iff b is in S and $x \preceq b$ for all $x \in S$;
- a is a minimal element of S iff a is in S and there is no $x \in S$ such that $x \prec a$;
- a is a minimum of S iff a is in S and $a \preceq x$ for all $x \in S$;
- the element b is an upper bound of S if $x \preceq b$, for each element $x \in S$;
- the element a is a lower bound of S if $x \succeq a$, for each element $x \in S$;
- the *meet* of a and b, denoted by $a \wedge b$, is the maximum of all lower bounds for the set $\{a, b\}$, i.e., $a \wedge b = max\{w \in P : w \preceq a, w \preceq b\}$, the greatest lower bound for a and b;
- the $join$ of a and b, denoted by $a \vee b$, is the minimum of all upper bounds for the set $\{a, b\}$, i.e., $a \vee b = min\{w \in P : a \preceq w, b \preceq w\}$, the least upper bound for a and b.

A subset S of a poset P is directed if every finite subset of S has an upper bound in S. The empty subset of P is not directed.

An upper set (also called an upward closed set or just an upset) of a partially ordered set $(P, \preceq)$ is a subset L with the property that, if x is in L and $x \preceq y$, then y is in L.

Let $(X, \leq)$ be an ordered set. We can define the Alexandroff topology $T(\leq)$ on X by choosing the open sets to be the upper sets. For any point $x \in X$ we put $x^+ = \{y \in X : x \leq y\}$. The set U is open in the Alexandroff topology if and only if $U = \cup\{x^+ : x \in U\}$.

Any T_0-topology generates the partial order $\leq_T$: $x \leq_T y$ if and only if $x \in cl_X\{y\}$ [6, 12]. We have $T(\leq_T) \subset T$. If (X, T) is an Alexandroff space, then $T(\leq_T) = T$ and vice versa.

Any ordering may be generated by the discrete quasi-metric and vice versa. Let $\leq$ be an ordering on a set X. We put $d_{\leq}(x, y) = 0$ if $x \leq y$ and $d_{\leq}(x, y) = 1$ if $x \not\leq y$. If d is a quasi-metric on X, then we put $x \leq_d y$ if and only if $d(x, y) = 0$. Obviously that $\leq_{d_{\leq}} = \leq$ and $d_{\leq_d} = d$ for any discrete quasi-metric d.

Distinct poset structures have been introduced to accommodate the needs of information theories. In the 1960's, Dana Scott introduced continuous lattices [170, 171, 172, 174, 175, 173] into computer science as a means of providing mathematical models for a system of types that justify recursive definitions of these types. In time, the order theoretic models Scott and others considered evolved into what we now call domains (see [1, 101, 182]). The level of abstraction required to understand domain theory remained an obstacle to its widespread use. To remedy this problem, Scott imported from logic the notion of an information system to provide a set-theoretic approach to domains [175]. In this setting, every information system gives rise to a domain in a canonical way. The Hoare powerdomain is an order-theoretic analog of the power set and is used in programming semantics as a model for angelic nondeterminism (see, for example, Plotkin [162]). Theory of lattices with theory of fuzzy sets were applied in information theory by G. Ciobanu, C. Vaideanu and A. Alexandru [10, 73].

A poset P is said to be directed-complete if the join of every directed subset of P exists in P. A subset S of poset P is a down-set of P provided $S = \{p \in P : p \leq a \textit{ for some } a \in S\}$. A down-set of P is Scott-closed if it contains the join of each of its directed subsets. An element x of a P is compact if, whenever x is below the supremum of a directed subset set S of P, then $x \in \{p \in P : p \leq a \textit{ for some } a \in S\}$. We use $K(P)$ to denote the subposet of compact elements of P. A directed-complete poset P is algebraic if, for all $p \in P$, the set $K(p) = \{x \in P : x \leq p\} \cap K(P)$ is directed and $p = \vee K(p)$. We use the term "domain" for an algebraic poset in which the meet of every non-empty subset exists. We will let $\Gamma(P)$ denote the set of all Scott-closed subsets of the directed-complete poset P, ordered by set-inclusion. It is easy to see that $\Gamma(P)$ is closed with respect to finite set-unions and arbitrary set-intersections. Hence $\Gamma(P)$ is the family of closed sets for a topology on P, called the Scott topology on P. The lattice of non-empty Scott-closed subsets of a domain D is called the Hoare powerdomain of D [107].

A domain representation of a topological space X is a function, usually a quotient map, from a subset of a domain onto X (see [35]). The theory of domains was improved by Yu. L. Ershov [88, 90, 89, 91] and now is called the Scott - Ershov theory of domains.

Definition 1.3.2. ([107]). An information system is a triple $\mathcal{S} = (S, Con, \vdash)$ consisting of:

1. a set S whose elements are called propositions or tokens;

2. a non-empty subset Con of the set of all finite subsets $Fin(S)$ of a set S, called the consistency predicate;

3. a binary relation $\vdash$ on Con, called the entailment relation.

   These entities satisfy the following axioms:
   $(IS1)$. Con is a down-set S of $Fin(S)$ – with respect to set-inclusion – such that $\cup Con = S$.
   $(IS2)$. if $A \subset Con$ and $B \subset A$, then $A \vdash B$.
   $(IS3)$. if $A, B, C \in Con$, $A \vdash B$, and $B \vdash C$, then $A \vdash C$.

$(IS4)$. if $A, B, C \in Con$, $A \vdash B, and A \vdash C$, then $B \cup C \in Con$ and $A \vdash (B \cup C)$.

Axiom $(IS1)$ implies that every singleton subset of S is a member of Con and that whenever $A \in Con$ and $B \subset A$, then $B \in Con$. Axioms $(IS2)$ and $(IS3)$ imply that $(Con, \vdash)$ is a preordered set, that is, $\vdash$ is a reflexive and transitive relation on Con. The above definition of an information system is different from the definitions of Scott [175], Davey and Priestly [78], Droste and Göbel [83].

1.4. Universal topological algebras

The notion of universal algebra has been introduced in the book of Alfred North Whitehead "A Treatise on Universal Algebra", published in 1898 [198], with the goal to expand and unify algebraic structures such as:

1. The fields of real and complex numbers.
2. Field of hyperbolic complex numbers introduced by James Cockle in 1848.
3. Lie algebras introduced by Sophus Lie during 1870 - 1874. The term Lie algebra was introduced by Hermann Weyl in 1930.
4. Non-commutative field of hypercomplex numbers (quaternions) introduced by William Rowan Hamilton in 1843, and earlier by Leonard Euler and Benjamin Olinde Rodrigues.
5. Non-commutative and non-associative field of octonions discovered by John Graves and Arthur Cayley in 1843.
6. Boolean algebra and logic algebra introduced in 1847 by George Boole and Augustus De Morgan respectively, who contributed to reforming mathematical logic.
7. Matrix algebra introduced by James Joseph Sylvester.
8. Algebra of hyperbolic quaternions introduced by Alexander Macfarlane in 1890.

Whitehead wrote in his book: "Such algebras have an intrinsic value for a separate detailed study; also they are worthy of comparative study, for the sake of the light thereby thrown on the general theory of symbolic reasoning, and on algebraic symbolism in particular. The comparative study necessarily presupposes some previous separate study, comparison being impossible without knowledge." Whitehead's work was too early and was not appreciated in his time. The study of universal algebras intensified after 1930, thanks to the works of Garrett Birkhoff and Oystein Ore, in which the study of diverse classes of abstract algebras, closure relations, Galois connections, lattice theory and graph theory were initiated.

Between 1935 and 1950, Birkhoff introduced varieties and quasivarities of universal algebras, free algebras, universal algebra congruence, subalgebra lattice, homomorphism theorems. Due to the second world war, the results published by Anatol Maltsev in the years 1938 – 1946 were not noted until the early 50s of the last century. Alfred Tarski's plenary lecture in 1950 at the Cambridge International Mathematics Congress inaugurated a new era. After 1950, various aspects of model theory were studied, with uncommon applications in mathematical logic, language theory, automata theory, with contribution of the following mathematicians: A. Robinson, A. Tarski, G. Birkhoff, C.C. Chang, L. Henkin, S. C. Kleene, B. Jonsson, A. Church, S. Eilenberg, S. MacLane, R. Lyndon, A. I. Maltsev, V. I. Arnautov, M. A. Arbib, V. M. Gluşkov, N. Chomsky, M. Minsky, S. Ginsburg, D. Scott, D. A. Huffman, E. Marczewski, J. Mycielski, P. J. Higgins, B. I. Plotkin, Yu. I. Manin, S. Marcus, A. G. Kurosh, V. I. Glivenko, V. D. Belousov, A. P. Ershov, O. B. Lupanov, A. D. Wallace and others (see [11, 13, 30, 17, 18, 19, 32, 34, 74, 76, 78, 86, 88, 90, 96, 97, 115, 116, 131, 133, 134, 144, 145, 165, 194]). The study of topological algebras was initiated with the study of Lie groups, topological groups and topological linear spaces.

Whitehead's definition has practically not changed. An universal algebra is a set A together with a collection of operations on A. An n-ary operation on A is a function that takes n elements of A and returns a single element of A. Thus, a 0-ary operation (or nullary operation) can be represented simply as an element of A, or a constant, often denoted by a letter like a. All topological universal algebras of the same type one can construct in the following way (see [51, 54, 50]):

1. We fix a sequence $\{E_n : n \in \omega = \{0, 1, 2, ...\}\}$ of topological spaces. The space E_n is declared the space of n-ary operation symbols.

2. The discrete sum E of the spaces $\{E_n : n \in \omega\}$ is the signature (type, language). If the space E is discrete, then we say that E is a discrete signature.

3. A topological algebra of signature E is a family $(G, \{e_n : E_n \times G^n \longrightarrow G : n \in \omega\})$, where G is a non-empty space and $e_n : E_n \times G^n \longrightarrow G$ is a continuous mapping for each $n \in \omega$. The space G is called the underlying space and for all $n \in \omega$ and $p \in E_n$ we define the n-ary operation $p : G^n \longrightarrow G$, where $p(z) = e_n(p, z)$ for each $z \in G^n$.

4. Now we say that E is the space of fundamental operation symbols.

Definition 1.4.1. Let G be an algebra, subset $A \subseteq G$ is called subalgebra, if $A \neq \emptyset$ and $u_{(n,G)}(E_n \times A^n) \subseteq A$ for all n. In this case, it is considered that $u_{(n,A)} = u_{(n,G)}|E_n \times A^n$.

Definition 1.4.2. Let A, B be two universal E-algebras. The mapping $\varphi : A \to B$ is called homomorphism, if

$$\varphi(u_{(n,A)}(f, x_1, x_2, ..., x_n)) = u_{(n,B)}(f, \varphi(x_1), \varphi(x_2), ..., \varphi(x_n))$$

for any n, $f \in E_n$ şi $x_1, x_2, ..., x_n \in A$.

If homomorphism is bijective, then it is called isomorphism.

Two isomorphic algebras differ only by the nature of the elements. Because of this, two isomorphic algebras are identified.

Let A, B be two universal quasi-topological E-algebras. The homomorphism $\varphi : A \longrightarrow B$ can be a:

- continuous homomorphism;
- continuous isomorphism;
- isomorphism and homeomorphism, called topological isomorphism.

We identify topologically isomorphic topological algebras. We consider that any topological space is a T_{-1}-space. We fix continuous signature E composed of subspaces $E_0, E_1, \ldots$ with $-1 \leq i \leq 3, 5$.

We denote by $A(E)$ totality of universal E-algebras, by $K(E, i)$ the totality of universal topological E-algebras which are T_i -spaces.

If $n \in \omega$, $f \in E_n$ and $G \subseteq Q(E, i)$, then the n-ary operation f is determined by $f : G^n \longrightarrow G$, where $f(x_1, x_2, ..., x_n) = u_{(n,G)}(f, x_1, x_2, ..., x_n)$. These are base (initial) operations.

Derived operations (terms) are determined as compositions of initial operations. For this purpose on the class of E-algebras, we add the following operations:

- $E'_0 = E_0$;
- $E'_1 = E_1 \oplus \{p^1_1\}$, where $p^1_1 \notin E$, and $p^1_1(x) = x$ for any $G \in A(E)$ and any $x \in G$;
- $E'_2 = E_2 \oplus \{p^2_1, p^2_2\}$, where $p^2_1, p^2_2 \notin E$, and $p^2_1(x, y)\ x$ and $p^2_2(x, y) = y$ for any $G \in A(E)$ and any $x, y \in G$;
- $E'_n = E_n$ for any $n \in \omega$ and $n \geq 3$;
- $E' = \cup\{E'_n : n = \omega\}$.

Definition 1.4.3. The set $T(E)$ of algebraic operations defined on algebras from $A(E)$ is the totality of derived operations with the following properties:

1. $E' \in T(E)$ *for any n*;
2. if $n \geq 1$, $f \in E'_n$, $g_1, g_2, ..., g_n \in T(E)$, then $f(g_1, g_2, ..., g_n) \in T(E)$;
3. the are no other operations in $T(E)$.

For each term its arity is determined: $n \geq 1, f \in E_n$, $g_1...m_1 : g_2...m_2; ...; g_n...m_n$, then $f(g_1, g_2, ..., g_n)$ will have $m_1 + m_2 + ... + m_n$ arity.

Definition 1.4.4. The totality $P(E)$ of polynomials is the smallest set of algebraic operations with the following properties:

1. $T(E) \subseteq P(E)$;
2. if $n \geq 1$, $g \in T(E)$ is a n-ar term, $n \geq m \geq 1$, $p : \{1, 2, ..., n\} \rightarrow \{1, 2, ..., m\}$ is a mapping, then $g(x_{p(1)}, x_{p(2)}, ...x_{p(n)}) \in P(E)$;
3. $P(E)$ does not contain any other operations.

Definition 1.4.5. Let $p(x_1, x_2, ..., x_n)$ and $q(y_1, y_2, ..., y_m)$ be two polynomials. The expression

$$p(x_1, x_2, ..., x_n) = q(y_1, y_2, ..., y_m)$$

is called an algebraic identity.

Let I be a set of identities, then some algebras $G \in A(E)$ satisfy these identities, and other do not. Algebras satisfying the identities are denoted by $A(E, I)$. Such classes of universal E-algebras are called *variety* or *primitive class*.

Definition 1.4.6. A non-empty class $\mathcal{V}$ of topological E-algebras is called a quasivariety of topological E-algebra, if it satisfies the following conditions:
(Σ) for any subalgebra B of some algebra $A \in \mathcal{V}$ we have $B \in \mathcal{V}$;
(Π) if $\{G_\alpha : \alpha \in A\} \subset \mathcal{V}$, then $\Pi\{G_\alpha : \alpha \in A\} \in \mathcal{V}$.

Definition 1.4.7. Universal algebra with a single binary algebraic operation is called a grupoid.

Binary operation $(\cdot)$ on algebra G can be:

- commutative: $x \cdot y = y \cdot x$ for any $x, y \in G$;
- associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for any $x, y, z \in G$.

Let $(G, \cdot)$ be a grupoid with $e \in G$. The element e is called the identity element (or neutral) in G, if $e \cdot a = a \cdot e = a$, for any $a \in G$. The identity element can can be defined for any operation with arity ≥ 2. The element $e \in G$ is called the identity (or neutral) element in G for n-ary operation $u : G^n \longrightarrow G$, $n \geq 2$, if for any $i \leq n$ we have $p(x_1, x_2, ..., x_n) = x_i$ given that $x_j = e$ for any $j \neq i$. The existence of the identity element is important in the theory of universal topological algebras.

Definition 1.4.8. Semigroup is a grupoid $(G, \cdot)$ that satisfies the identity $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

Definition 1.4.9. Monoid is a semigroup with an identity element.

Definition 1.4.10. Group is a universal algebra G with a binary operation $\{\cdot\}$, a unary operation $\{^{-1}\}$ and a nullary operation that fixes an element e with the following identities:

1. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;

2. $x \cdot x^{-1} = x^{-1} \cdot x = e$;

3. $e \cdot x = x \cdot e = x$.

The notion of group appeared in the nineteenth century in researches related to geometry and permutations. In geometry, they appeared as groups of geometric transformations: the group of isometries, the group of similarities, the group of affine transformations, the group of projective transformations, topological transformation group (continuous). This fact unites topology with geometry. The permutations group is a group of transformations of a finite set. Evariste Galois applied the subgroups of the permutations group to solve the problem of solving equations in radicals. These ideas led to the creation of the Galois theory, an important field of contemporary mathematics. Transformation groups have also influenced research in physics: Lorentz transformations, Einstein's relativity theory, etc.

The study of semigroups is connected to the study of algebraic structures with more complex axioms such as groups or rings. The first use of the term belongs to J.-A. de Seguier [176] in 1904. An impulse of research in the field of semigroups was determined by its applications in information theory and automata theory [5, 11, 86, 96, 127, 100, 142]. Algebraic and topological structures are important in the theoretical study of problems related to automata and information theory [2, 3, 11, 86, 132, 137, 136, 140, 100, 143, 181, 195, 200, 117, 33, 47, 131, 139, 141, 142, 180, 181].

Regarding the general theory of semigroups we mention Clifford and Preston [74], elementary information regarding universal algebra - Gratzer [97] and Choban [51, 50, 54], and for fundamental information about automata theory - Hopcroft and Ullman [117].

A topological quasigroup is a non-empty space G with three binary continuous operations $\{\cdot, r, l\}$ and identities $x \cdot l(x, y) = r(y, x) \cdot x = l(x, x \cdot y) = l(r(x, y), x) = r(y \cdot x, x) = y$ (see [30]).

A topological bigroupoid is a topological space G with two binary continuous operations $\{\cdot, *\}$ for which there exists an element $e \in$G such that $x \cdot e = x$ for each $x \in G$. A bigroupoid G is a bigroupoid with a division or, briefly a d-bigroupoid if for each two elements $a, b \in G$ there exist two elements $c, p \in G$ such that $a \cdot c = b$ and $p \cdot a = b$. A bigroupoid G is called an a-bigroupoid if $x \cdot (y * z) = (x * y) \cdot z$ for all $x, y, z \in G$.

One of the general problems in topological algebra is determined by the study of the relationships between topological properties of the spaces and underlying algebraic structures on them. That general problem is examined in the light of the following three problems:

Problem DT. *Let G be an E-algebra. Determine the kinds of topologies, which can be considered on the E-algebra G that makes it a topological E-algebra.*

Problem DA. *Let G be a topological space. Determine the types of algebraic structures that can be considered on the space G, which makes it a topological E-algebra.*

Problem DC. *Research of applications of the Theory of Topological Algebras*.

One of the general problems, determined by the direction of the posed problem DA, is the following:

Problem DAT. *Let G be a topological non-empty space, E be a signature and Λ be a set of identities. Is it true that G admits a structure of topological E-algebra for which G is a topological E-algebra with the identities Λ?*

One of the first results in this direction is the Pontryagin version of the Frobenius theorem in the abstract algebra (see [185, 186, 66]).

TFP(Frobenius - Pontryagin). *Let D be a connected locally compact division ring. Then:*

1. *If D is associative and commutative, then either D is the ring of reals $\mathbb{R}$, or the ring $\mathbb{C}$ of complex numbers.*
2. *If D is associative and non-commutative, then D is the ring of quaternions $\mathbb{H}$.*
3. *If D is non-associative, then D is the ring of octonions $\mathbb{D}$.*

The algebra of quaternions was discovered by Hamilton in 1843 and the algebra of the octonions - by J. T. Graves in 1843. The Cayley-Diskson construction produces a sequence of topological algebras over the given topological field (in particular over the reals). In the case of reals, we obtain the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{D}$ (see [24]). Indeed, let R be a topological ring with involution $x \to x^*$. Denote by $A(R,^*)$ the set $R^2 = R \times R$ with the operations:

- $(x, y) + (u, v) = (x + u, y + v)$;
- $(x, y) \cdot (u, v) = (xu - v^* y, vx + yu^*)$;
- $(x, y)^* = (x^*, -y)$.

Then $A(R,^*)$ is a topological ring with the involution and a topological R-module. The mapping $x \longrightarrow (x, 0)$ is the natural embedding of the ring R into $A(R,^*)$. As a rule, the point $x \in R$ is identified by the point $(x, 0) \in A(R,^*)$ and one may consider that $R \subset A(R,^*)$.

If on the field $\mathbb{R}$ of reals the identical mapping $x \longrightarrow x^* = x$ is the given involution, then $\mathbb{C} = A(\mathbb{R},^*)$ is the algebra of complex numbers, $\mathbb{H} = A(\mathbb{C},^*)$ is the algebra of quaternions (hypercomplex) number and $\mathbb{D} = A(\mathbb{H},^*)$ is the algebra of octonions. The algebras $\mathbb{H}_1 = A(\mathbb{H},^*)$ and $\mathbb{H}_{n+1} = A(\mathbb{H}_n,^*)$ relatively to the multiplication are not with division for all $n \geq 1$.

Corollary. *Let G be an infinite connected and locally compact space. If $dimG \notin \{1, 2, 4, 8\}$, then G does not admit the structure of the topological division ring*.

Obviously, any topological space G admits structures of topological E-algebras. It is sufficient to fix some continuous mapping $e_n : E_n \times G^n \longrightarrow G$ for each $n \in \omega$. In particular, the operation $xy = x$ determines on G the structure of a topological semigroup with a right identity: the element $e \in G$ is a right (respectively, left) identity if $xe = x$ (respectively, $ex = x$) for any $x \in G$.

Remark. There exists a metrizable connected compact space A such that if xy is a structure of a topological groupoid with right identity, then $xy = x$ for all $x, y \in A$. In this case any continuous mapping $f : A \times A \longrightarrow A$ is one of the projections or a constant mapping. The space A is called the Cook continuum (see [185, 186]).

1.5. Spaces of strings. Languages

Fix a non-empty set A. The set A is called an alphabet. Let $L^*(A)$ be the set of all finite strings $a_1a_2\ldots a_n$ with $a_1, a_2, \ldots, a_n \in A$. Let ε be the empty string. Consider the strings $a_1a_2\ldots a_n$ for which $a_i = \varepsilon$ for some $i \leq n$. If $a_i \neq \varepsilon$, for any $i \leq n$ or $n = 1$ and $a_1 = \varepsilon$, the string $a_1a_2\ldots a_n$ is called a *irreducible string* or *canonical string*. The set $Sup(a_1a_2\ldots a_n) = \{a_1, a_2, \ldots, a_n\} \cap A$ is the support of the string $a_1a_2\ldots a_n$ and $l(a_1a_2\ldots a_n) = |\{i \leq n : a_i \neq \varepsilon\}|$ is the length of the string $a_1a_2\ldots a_n$. For two strings $a_1\ldots a_n$ and $b_1\ldots b_m$, their product (concatenation) is $a_1\ldots a_nb_1\ldots b_m$. If $n \geq 2$, $i < n$ and $a_i = \varepsilon$, then the strings $a_1\ldots a_n$ and $a_1\ldots a_{i-1}a_{i+1}\ldots a_n$ are considered equivalent. In this case any string is equivalent to one unique canonical string. We identify the equivalent strings. The set $L(A)$ of all canonical strings is the class of equivalent strings. In this case $L^*(A)$ is a semigroup and $L(A)$ becomes a monoid with identity ε. The set $L(A)$ is not a subsemigroup of $L^*(A)$. Only the set $L(A) \setminus \{\varepsilon\}$ is a subsemigroup of the semigroup $L^*(A)$.

Let $\bar{A} = A \cup \{\varepsilon\}$, $Sup(a, b) = Sup(a) \cup Sup(b) \cup \{\varepsilon\}$, and $Sup(a, a) = Sup(a) \cup \{\varepsilon\}$. It is well known that any subset $L \subset L(A)$ is an abstract language over the alphabet A.

Fix an alphabet A and let $\bar{A} = A \cup \{\varepsilon\}$. We assume that $\varepsilon \in \bar{A} \subseteq L(A)$. Let a, b be two strings.

We put $A^{-1} = \{a^{-1} : a \in A\}$, $\varepsilon^{-1} = \varepsilon$, $(a^{-1})^{-1} = a$ for any $a \in A$ and consider that $A^{-1} \cap \bar{A} = \emptyset$. Denote $\check{A} = A \cup A^{-1} \cup \{\varepsilon\}$. Let $\check{L}(A) = L^*(\check{A})$ be the set of all strings over the set $\check{A}$ with the empty string ε. The strings over the set $\check{A}$ are called *words*. A word $a = a_1a_2 \cdots a_n \in \check{L}A)$ is called an irreducible string if $n = 1$ and $a_1 \in \check{A}$, or $n \geq 2$, $a_i \neq \varepsilon$ for any $i \leq n$ and $a_j^{-1} \neq a_{j+1}$ for each $j < n$.

Let $a = a_1a_2 \cdots a_n \in \check{L}(A)$ and $n \geq 2$. Then:

- if $i \leq n$ and $a_i = \varepsilon$, then the words $a_1a_2...a_n$ and $a_1\ldots a_{i-1}a_{i+1}\ldots a_n$ are considered equivalent;

- if $i < n$ and $a_i^{-1} = a_{i+1}$ then the words $a_1a_2...a_n$ and $a_1\ldots a_{i-1}\varepsilon a_{i+2}\ldots a_n$ are considered equivalent.

In this case any word $a_1 a_2 \cdots a_n \in \check{L}(A)$ is equivalent to one unique irreducible word from $\check{L}(A)$. We identify the equivalent words. The classes of equivalence form the free group $F(A)$ over A with the identity ε. We have that $L(A)$ is a submonoid of the group $F(A)$.

Let $a = a_1 a_2 ... a_n \in F(A)$ be an irreducible word. The representation $a = x_1 x_2 ... x_m \in L^*(A)$ is called an *almost irreducible* representation of a if there exists a sequence $1 \leq i_1 < i_2 < ... < i_n \leq m$ such that $a_j = x_{i_j}$ for any $j \leq n$ and $x_i = \varepsilon$ for each $i \in \{1, 2, ..., m\} \setminus \{i_1, i_2, ..., i_n\}$. If $a = a_1 a_2 ... a_n \in L^*(A)$ is a representation of the string a, then $a_1 a_2 ... a_n$ is an almost irreducible word.

If $a = a_1 a_2 ... a_n$, then $a^s = a_n a_{n-1} ... a_2 a_1$ and $a^{-1} = a_n^{-1} a_{n-1}^{-1} ... a_2^{-1} a_1^{-1}$. The word a^s is the symmetric word of a and a^{-1} is the inverse word of a. If a and b are equivalent words, then the words a^{-1} and b^{-1} are equivalent and the words a^s and b^s are equivalent too. Hence the mappings $\cdot^s, \cdot^{-1} : F(A) \longrightarrow F(A)$ are the group automorphisms. Obviously that $L(A)^s = L(A)$.

There exists a unique semigroup homomorphism $\pi_A : \check{L}(A) \longrightarrow F(A)$ such that $\pi_A(a) = a$ for each $a \in \check{A}$. In this case $\pi_A(a) = a$ for each irreducible word $a \in F(A) \subset \check{L}(A)$.

Fix a distance d on $\bar{A}$. We put:

- $d_H(a, b) = d(a, b)$ for any $a, b \in \bar{A}$;
- $d_H(a^{-1}, b^{-1}) = d(b, a)$ for any $a, b \in \bar{A}$;
- $d_H(a, b^{-1}) = d(a, \varepsilon) + d(\varepsilon, b^{-1})$ and $d_H(a^{-1}, b) = d(a^{-1}, \varepsilon) + d(\varepsilon, b)$ for any $a, b \in \bar{A}$;
- $d_H(a_1 a_2 \cdots a_n, b_1 b_2 \cdots b_m) =$

$\Sigma\{d_H(a_i, b_i) : i \leq min\{n, m\}\} + \Sigma\{d_H(a_i, \varepsilon) : n < i \leq m\} + \Sigma\{d_H(\varepsilon, b_j) : m < j \leq n\}$

for any two words $a_1 a_2 \cdots a_n, b_1 b_2 \cdots b_m \in \check{L}(A)$.

We say that d_H is the Hamming distance between two words generated by d, and d_H is the extension of d on $\check{L}(A)$.

Distance d generates on the free group $F(A)$ the Graev - Markov distance d_G:

$$d_G(a, b) = min\{d_H(a', b') : a', b' \in \check{L}(A), a = \pi_A(a'), b = \pi_A(b')\}$$

for any $a, b \in F(A)$.

Hamming [106] considered the distance d_H on $L^*(A)$ for the discrete metric d on $\bar{A}$. The function distance d_H has the following properties:

1. d is a distance if and only if d_H is a distance;

2. d is a metric if and only if d_H is a metric;

3. d is a qvasimetric if and only if d_H is a quasi-metric.

Graev [98] considered d_G for a metric and proved:

4. d is a metric if and only if d_G is a metric;

5. For a metric d we have that d_G is an invariant metric and $d(a,b) = d_G(a,b)$ for any $a, b \in \bar{A}$.

The condition that d is metric was essential in the Graev constructions. The topology generated by an invariant metric on group is a group topology, but the topology generated by an invariant quasi-metric on group is not always a group topology. Moreower, d_G may be not a distance for some distance d.

Example 1.5.1. *Let G be the additive group of real numbers, $d(x, y) = min\{1, y - x\}$ if $x \leq y$ and $d(x, y) = 1$ if $y < x$. Then:*

- d is an invariant qvasimetric on G;

- $(G, T(d))$ is the Sorgenfrey line and is not a metrizable space;

- $(G, T(d))$ is a topological semigroup and is not a topological group.

Example 1.5.2. *Let $A = \{0, 1, 2\}$, $d(x, x) = 0$ for any $x \in \bar{A}$, $d(0, 1) = d(1, 2) = d(2, 0) = d(a, \epsilon) = d(b, \varepsilon) = d(\varepsilon, c) = 0$ and $d(1, 0) = d(2, 1) = d(0, 2) = d(\epsilon, a) = d(\varepsilon, b) = d(c, \varepsilon) = 1$. Then d is a discrete distance on $\bar{A}$, where $a, b, c \in A$. Fix $a = 0$ and $b = 2$ from $L(A) \subset F(A)$. Then $a\varepsilon$ and $\varepsilon 2$ are almost irreducible representation of a and b. We have $\pi_A(0) = \pi_A(0\varepsilon) = a$ and $\pi_A(2) = \pi_A(\varepsilon 2)$ $= b$. Since $d_H(2, 0) = d_H(0\varepsilon, \varepsilon 2) = 0$, we have $d_G(a, b) = d_G(b, a) = 0$. Hence d_G is not a distance on $L(A)$ and $F(A)$.*

Hence, it is natural to have the following open question: under which conditions d_G is a distance on $L(A)$?

For the discrete metric d on $\bar{A}$ the metric d_G coincides with the Levenstein metric on $L(A)$.

The V. I. Levenshtein's distance $d_L(a, b)$ between two strings $a = a_1 a_2 \cdots a_n$ and $b = b_1 b_2 \cdots b_m$ from $L(A)$ is defined as the minimum number of insertions, deletions, and substitutions required to transform one string to the other [130, 55, 56, 57]

1.6. Free algebras. Maltsev's Problems

Fix a continuous signature $E = \oplus\{E_n : n = 0, 1, 2, ...\}$ and a quasivariety $\mathcal{V}$ of topological E-algebras that satisfy the following relations:

1. Topological non-triviality: There exists a topological algebra $G \in \mathcal{V}$ which contains a non-proper open subset U ($\emptyset \neq U \neq G$).

2. Topological completeness: If $(G, T_0) \in \mathcal{V}$ and T is a T_0-topology on G such that (G, T) is a topological E-algebra, then $(G, T) \in \mathcal{V}$.

In [51, 133] (see also [52, 53, 54, 50]) was proved: For each non-empty topological space X there exist two topological E-algebras $F(X, \mathcal{V}) \in \mathcal{V}$ and $F^o(X, \mathcal{V}) \in \mathcal{V}$ and a continuous mapping $v_X : X \longrightarrow F^o(X, \mathcal{V})$ with the following properties:

1. The set $v_X(X)$ generates the algebra $F^o(X, \mathcal{V})$.

2. If $g : X \longrightarrow G \in \mathcal{V}$ is a continuous mapping, then there exists a unique continuous homomorphism $\bar{g} : F^o(X, \mathcal{V}) \longrightarrow G$ such that $g = \bar{g} \circ v_X$.

3. X is a subset of the E-algebra $F(X, \mathcal{V})$ and the set X generates the algebra $F(X, \mathcal{V})$.

4. If $g : X \longrightarrow G \in \mathcal{V}$ is a mapping, then there exists a unique continuous homomorphism $\bar{g} : F^o(X, \mathcal{V}) \longrightarrow G$ such that $g = \bar{g}|X$.

5. There exists a unique continuous homomorphism $w_X : F(X, \mathcal{V}) \longrightarrow F^o(X, \mathcal{V})$ such that $v_X = w_X|X$.

The algebra $F(X, \mathcal{V})$ is called the free E-algebra on the space X in the class $\mathcal{V}$ and the pair $(F^o(X, \mathcal{V}), v_X)$ is called the topological free E-algebra on the space X in the class $\mathcal{V}$. For any space X the free objects are unique.

In 1957 A. I. Maltsev [133] posed the following problems:

First Maltsev's Problem: Under which conditions the mapping v_X is an embedding?

Second Maltsev's Problem: Under which conditions the homomorphism w_X is a continuous isomorphism?

For complete regular spaces X the Maltsev Problems were solved affirmatively by S. Swierczkowski [184] in the case of discrete signature E, and by M. M. Choban and S. S. Dumitrashcu for any signature [84, 51].

Since any topological group is a completely regular space, this result has a definitive character. There are various quasivarieties and varieties that contain T_0-spaces that are not completely regular. For such cases Maltsev's problems remain open! For example, the variety of topological monoids contains T_0-monoids and for quasivarieties of topological monoids these problems are not solved.

Maltsev's problems are related to to the following problem:

Topologization Problem. Let G be an E-algebra. What types of topologies T exists on G for which (G, T) is a topological algebra?

Particular cases of the Topologization problem are:

Problem PT1. Under what conditions E-algebra G is topologizable with a T_0-topology, or with a Hausdorff topology, or with a metrizable topology, or with a non-discrete quasi-metrizable topology?

Problem PT2. Under what conditions E-algebra G admits compact topologizations?

Problem PT3. Let X be a quasi-metrizable space. Under what conditions there exists a quasi-metrizable topologization on $F(X, \mathcal{V})$ for which X is a subspace?

Regarding the PT1 and PT2 problems, profound results were obtained by S. Hartman, J. Mycielsky, V. I. Arnautov, M. I. Ursul, P. Chircu. For metrics the PT3 problem was solved by M.

I. Graev [98], in the case of the variety of all groups, and by M. M. Choban [52] in the case of discrete signature. For this purpose M. M. Choban [52] introduced the notion of stable distance on a universal algebra. The case of semigroups remained open.

L. Pontryagin, in an unpublished letter to A. Weil, proved that every topological group is completely regular. This result was used by A. Weil [197] in the theory of uniform spaces. Since normality is the next after complete regularity interesting property of separability of spaces, it is natural to raise the question of whether every topological group is a normal space. A. Markov [135] managed to solve this question in a negative sense, proving the following theorem:

Theorem TM1. *Any completely regular space can be topologically embedded into a linear topological locally convex space as a closed subset of the latter.*

The notion of locally convex linear space was introduced by A. Kolmogoroff (see [124, 26]). Using theorem TM1, we can build a topological group which is not a normal topological space, in the following way. Let X be some completely regular, but not normal space. Such spaces exist, as shown by A. N. Tikhonov in [191].

By virtue of TM1 Theorem, there exists a linear topological locally convex space S, topologically containing X as a closed subset. Space S cannot be a normal space, because otherwise all its closed subsets would be normal too. Since every linear topological space is an abelian topological group, S is an example of abnormal topological group. The proof of the theorem is based on certain ideas of A. Weil [197] and on the result of D.H. Hyers [118], related to the construction of linear topological spaces by means of "pseudo-norms".

Let X be a subset of a topological group G. We say that X generates G if there exists no proper subgroup of G containing X.

Using the idea of pseudo-norms A. A. Markov [135] demonstrates the following two theorems.

Theorem TM2. *Let X be a completely regular space. Then there exists a topological group $F(X)$ with the following properties:*

1. *X is a closed subspace of $F(X)$;*
2. *X generates $F(X)$;*
3. *For any continuous mapping φ of X into any topological group G, there exists a continuous homomorphism Φ of the topological group $F(X)$ into the topological group G such that $\Phi(x) = \varphi(x)$ for every point x of X.*

Theorem TM3. *Let X be a completely regular space. Then there exists a topological Abelian group $A(X)$ with the following properties:*

1. *X is a closed subspace of $A(X)$;*
2. *X generates $A(X)$;*

3. *For any continuous mapping φ of X into any topological Abelian group G, there exists a continuous homomorphism Φ of the topological group $A(X)$ into the topological group G such that $\Phi(x) = \varphi(x)$ for every point x of X.*

At the end of his paper [135] A. Markov formulates the following problems:

Problem 1M. *To prove or to refute the assertion: the free topological groups of two completely regular spaces are topologically isomorphic, if and only if these spaces are homeomorphic.*

Problem 2M. *To prove or to refute the analogous assertion on free abelian topological groups.*

Problem 3M. *To prove or to refute the assertion: every uncountable group admits a non-normal topology.*

The topological group G is trivial if every it subset is open.

Problem 4M. *To prove or to refute the assertion: every infinite group admits a non-trivial topology.*

In 1948 M. Graev [98] solves negatively problems 1M and 2M. These negative solutions were the basis for studying the algebraic equivalence of topological spaces (see [13]). The problem of studying the common properties of M-equivalent spaces (in which the free groups are topologically equivalent) and A-equivalent spaces (in which the free Abelian groups are topologically equivalent) was formulated by L. Pontryagin in 1947.

Also, in 1948, M. Graev [98] significantly simplifies the proofs of theorems TM1, TM2 and TM3. Graev's main idea in the study of free groups was to replace the notion of "pseudo-norm" with the notion of "invariant psedo-metric". A continuous pseudo-metric on space X is a function d with the following properties:

$(P1)$. $d(x, y) \geq 0$ si $d(x, x) = 0$ for any $x, y \in X$;

$(P2)$. $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$;

$(P3)$. $d(x, y) = d(y, x)$ for any $x, y \in X$;

$(P4)$. $B(x, d, r) = \{y \in X : d(x, y) < r\}$ is an open set for any $r > 0$ and $x \in X$.

The X space is completely regular if and only if the space topology is generated by a family of pseudo-metrics. M. I. Graev examines spaces with point base (X, p), where $p \in X$. For the free group X of the set X, it is considered that $p \in X \subseteq F(X)$ and p is the neutral element of the $F(X)$ group.

M. Graev [98] proves the following two theorems:

Theorem TG1. *Let $F(X)$ be the free group of the set X, where X is a completely regular space. Then for any pseudo-metric d on the space X there exists an invariant pseudo-metric $\check{d}$ on $F(X)$ with the properties:*

1. *$d(x, y) = \check{d}(x, y)$ for any $x, y \in X$.*

2. *If $x, y \in F(X)$ are two distinct words, $x = x_1x_2...x_n$ and $y = y_1y_2...y_m$, where $x_1, x_2, ..., x_n$, $y_1, y_2, ..., y_m \in X \cup X^{-1}$, and $d(x_i, y_j) \neq 0$ for $x_i \neq y_j$, then $\check{d}(x, y) \neq 0$.*

3. *If d is a metric, then $\check{d}$ is also a metric.*

4. *If ρ is an invariant metric on $F(X)$ and $\rho(x, y) \leq d(x, y)$ for any $x, y \in X$, then $\rho(x, y) \leq \check{d}(x, y)$ for any $x, y \in F(X)$.*

Theorem TG2. *Let $A(X)$ be the free Abelian group of the set X, where X is a completely regular space. Then for any pseudo-metric d on the space X there exists an invariant pseudo-metric $\check{d}$ on $A(X)$ with the properties:*

1. *$d(x, y) = \check{d}(x, y)$ for any $x, y \in X$.*

2. *If $x, y \in A(X)$ are two different words, $x = x_1x_2...x_n$ and $y = y_1y_2...y_m$, where $x_1, x_2, ..., x_n$, $y_1, y_2, ..., y_m \in X \cup X^{-1}$, and $d(x_i, y_j) \neq 0$ for $x_i \neq y_j$, then $\check{d}(x, y) \neq 0$.*

3. *If d is a metric, then $\check{d}$ is also a metric.*

4. *If ρ is an invariant metric on $F(X)$ and $\rho(x, y) \leq d(x, y)$ for any $x, y \in X$, then $\rho(x, y) \leq \check{d}(x, y)$ for any $x, y \in A(X)$.*

These theorems were extended by M. M. Choban [52] for varieties of topological universal algebras with discrete signature.

Fix a discrete signature E, a quasivariety $\mathcal{V}$ of topological E-algebras and a space X.

Let $F(X, \mathcal{V})$ be the free E-algebra and X is included in $F(X, \mathcal{V})$ as generating subset. Denote $F_0(X) = X \cup e_0(E_0 \times G^0)$. Pseudo-metric d is stable on E-algebra G, if for any $n \geq 1$, any $u \in E_n$ and any $x_1, y_1, x_2, y_2, ..., x_n, y_n \in G$ we have $d(u(x_1, x_2, ..., x_n), u(y_1, y_2, ..., y_n)) \leq \Sigma\{d(x_i, y_i) : i \leq n\}$.

M. M. Choban [52] proves the following two theorems:

Theorem TC1. *Let $F(X, \mathcal{V})$ be the free E-algebra of the space X. Then for any pseudo-metric d on the set $F_0(X)$, where $d(x, y) \leq 1$ for any $x, y \in F_0(X)$, there exists a stable pseudo-metric $\check{d}$ on $F(X, \mathcal{V})$ with the properties:*

1. *$d(x, y) = \check{d}(x, y)$ for any $x, y \in F_0(X)$.*

2. *If $x, y \in F(X, \mathcal{V})$ are two distinct elements, u is an n-ary term , v is an m-ary term, $x = u(x_1, x_2, ..., x_n)$ and $y = v(y_1, y_2, ..., y_m)$, where $x_1, x_2, ..., x_n$, $y_1, y_2, ..., y_m \in F_o(X)$ and $d(x_i, y_j) \neq 0$ for $x_i \neq y_j$, then $\check{d}(x, y) \neq 0$.*

3. *If d is a metric, then $\check{d}$ is also a metric.*

4. *If ρ is a stable metric on $F(X, \mathcal{V})$, $\rho(x, y) \leq 1$ for any $x, y \in F(X, \mathcal{V})$ and $\rho(x, y) \leq d(x, y)$ for any $x, y \in F_0(X)$, then $\rho(x, y) \leq \check{d}(x, y)$ for any $x, y \in F(X, \mathcal{V})$.*

Theorem TC2. *Assume that there exists $n \geq 2$ and an n-ary term μ for which any E-algebra $G \in \mathcal{V}$ contains a neutral element. Let $F(X, \mathcal{V})$ be the free E-algebra of the space X. Then for any pseudo-metric d on set $F_0(X)$ there exists an invariant pseudo-metric $\check{d}$ on $F(X, \mathcal{V})$ with the properties:*

1. *$d(x, y) = \check{d}(x, y)$ for any $x, y \in F_0(X)$.*

2. *If $x, y \in F(X, \mathcal{V})$ are two distinct elements, u is an n-ary term, v is an m-ary term, $x = u(x_1 x_2 ... x_n)$ and $y = v(y_1 y_2 ... y_m)$, where $x_1, x_2, ..., x_n, y_1, y_2, ..., y_m \in F_o(X)$ and $d(x_i, y_j) \neq 0$ for $x_i \neq y_j$, then $\check{d}(x, y) \neq 0$.*

3. *If d is a metric, then $\check{d}$ is also a metric.*

4. *If ρ is an invariant metric on $F(X, \mathcal{V})$, $\rho(x, y) \leq 1$ for any $x, y \in F(X, \mathcal{V})$ and $\rho(x, y) \leq d(x, y)$ for any $x, y \in F_0(X)$, then $\rho(x, y) \leq \check{d}(x, y)$ for any $x, y \in F(X, \mathcal{V})$.*

Theorem TC2 contains Markov's and Graev's theorems.

Various aspects of the theory of topological groups are deeply reflected in the book [13]. Various results on topological algebras are contained in the works of [67, 108, 70, 161, 71, 23, 158, 159, 160, 105].

1.7. Conclusions for chapter 1

In the information theory and, particularly, in the theory of languages it is important the monoid $L(A)$ of all strings on the given alphabet A. The analysis of the Hamming and Livenshtein distances lead us to the problem of extension of the given distance ρ on A to an invariant distance ρ^* on $L(A)$. This problem is important and remains unsolved for any quasivariety of topological monoids. Since an invariant quasi-metric on $L(A)$ is a measure of similarity of information, it is important the following problem: the elaboration of methods to study the distances on free monoids, which contributes to obtaining effective methods of representation of information applicable in solving various distance problems.

For solving this problem it is important to study the following particular problems:

1. To determine the conditions of extension of given quasi-metric ρ on A and any quasivariety $\mathcal{V}$ to an invariant quasi-metric ρ^* on the free monoid $F^a(A, \mathcal{V})$.

2. To propose the algorithms of the calculation of the distance $\rho^*(a, b)$ between two information sequences $a, b \in L(A)$.

3. To determine the relations between topologo-geometrical properties of spaces (A, ρ) and $(L(A), \rho^*)$.

4. To propose methods of construction of weighted means and bisector sets of a given pair of strings.

5. To determine topologo-geometrical properties which are important in the analysis of information and image processing.

Hence, for solving the general problem, it is necessary to achieve the following objectives:

- to elaborate an effective method for extending the quasi-metrics on free monoids;
- to develop efficient mechanisms of information representation;
- to implement innovative algorithms to solve various problems related to text sequences, as well as geometrical aspects in the study of the information space;
- to describe the digital topologies on the discrete line.

2. EXTENSION OF QUASI-METRICS ON FREE TOPOLOGICAL MONOIDS

Chapter 2 presents main results obtained by the author during the research on the extension of quasi-metrics on free topological monoids.

The study material begins with the examination of the free topological monoids, followed by the construction method of the abstract free monoid. We discuss the properties of the Burnside quasivariety, and present the method of extending a quasi-metric on free monoid $F^a(X, \mathcal{V})$ for a non-Burnside quasivariety $\mathcal{V}$.

The results presented in this chapter successfully complement the works of other mathematicians in the domain of the distance extension on the abstract algebraic structures. More specifically, these results permit to solve the problems posed by A. I. Maltsev in 1958 for free universal topological problems. The author's work in this chapter is published in the articles [44, 57, 62] and serve as a base for research presented in the next chapters.

2.1. Free topological monoids

A class $\mathcal{V}$ of topological monoids is called a quasivariety of monoids if:

(F1) the class $\mathcal{V}$ is multiplicative;

(F2) if $G \in \mathcal{V}$ and A is a submonoid of G, then $A \in \mathcal{V}$;

(F3) every space $G \in \mathcal{V}$ is a T_0-space.

A class $\mathcal{V}$ of topological monoids is called a complete quasivariety of monoids if it is a quasivariety with the next property:

(F4) if $G \in \mathcal{V}$ and T is a T_0-topology on G such that (G, T) is a topological monoid, then $(G, T) \in \mathcal{V}$ too.

A quasivariety $\mathcal{V}$ of topological monoids is non-trivial if $|G| \geq 2$ for some $G \in \mathcal{V}$.

Let X be a non-empty topological space and $\mathcal{V}$ be a quasivariety of topological monoids. In the space X the base point $p_X \in X$ is fixed, i.e. any space is pointed.

A free monoid of a space X in a class $\mathcal{V}$ is a topological monoid $F(X, \mathcal{V})$ with the properties:

– $X \subseteq F(X, \mathcal{V}) \in \mathcal{V}$ and p_X is the unity of $F(X, \mathcal{V})$;

– the set X generates the monoid $F(X, \mathcal{V})$;

– for any continuous mapping $f : X \longrightarrow G \in \mathcal{V}$, where $f(p_X) = e$, there exists a unique continuous homomorphism $\bar{f} : F(X, \mathcal{V}) \longrightarrow G$ such that $f = \bar{f}|X$.

An abstract free monoid of a space X in a class $\mathcal{V}$ is a topological monoid $F^a(X, \mathcal{V})$ with the properties:

– X is a subset of $F^a(X, \mathcal{V})$, $F^a(X, \mathcal{V}) \in \mathcal{V}$ and p_X is the unity of $F^a(X, \mathcal{V})$;

– the set X generates the monoid $F^a(X, \mathcal{V})$;

– for any mapping $f : X \longrightarrow G \in \mathcal{V}$, where $f(p_X) = e$, there exists a unique continuous homomorphism $\hat{f} : F^a(X, \mathcal{V}) \longrightarrow G$ such that $f = \hat{f}|X$.

In the proof of the next assertion we use the Kakutani's method [121].

Theorem 2.1.1. *Let $\mathcal{V}$ be a non-trivial quasivariety of topological monoids. Then for each space X the following assertions are equivalent:*

1. *There exists $G \in \mathcal{V}$ such that X is a subspace of G and p_X is the neutral element in G.*
2. *For the space X there exists the unique free topological monoid $F(X, \mathcal{V})$.*

Proof. Implication $2 \to 1$ is obvious. Assume now that there exists $A \in \mathcal{V}$ such that X is a subspace of A and p_X is the neutral element in A. Let τ be an infinite cardinal number and $|X| \leq \tau$. Denote by $\mathcal{V}(\tau)$ the collection of all $G \in \mathcal{V}$ of the cardinality $\leq \tau$. Since we identify the topologically isomorphic topological monoids, the family $\mathcal{V}(\tau)$ is a set. Hence the collection $\{h_\mu : X \longrightarrow G_\mu : \mu \in M\}$ of all continuous mappings $f : X \longrightarrow G \in \mathcal{V}(\tau)$ with $f(p_X) = e \in G$ is a set too. Consider the diagonal product $h : X \longrightarrow G = \Pi\{G_\mu : \mu \in M\}$, where $h(x) = (h_\mu(x) : \mu \in M) \in G$ for every point $x \in X$. By construction, $h(p_X) = (e_\mu \in G_\mu : \mu \in M) = e \in G$ and h is a continuous mapping. Denote by $H(X)$ the submonoid of G generated by the set $Y = h(X)$ in G. For each $\eta \in M$ consider the projection $\pi_\eta : H(X) \longrightarrow G_\mu$, where $\pi_\eta(x_\mu : \mu \in M) = x_\eta$ for each point $(x_\mu : \mu \in M) \in H(X)$. Then $h_\eta = \pi_\eta \circ h$. Each projection π_η is a homomorphism.

Since $|Y| \leq |X| \leq \tau$, we have $|H(X)| \leq \tau$ and $H(X) \in \mathcal{V}(\tau)$.

For some $\lambda \in M$ we have that G_λ is a submonoid of A and $h_\lambda : X \longrightarrow G_\lambda$ is an embedding of X in G_λ and $e_\lambda = p_X$ is the unity of the monoid G_λ. We have $h_\lambda(x) = x$ for each $x \in X$. Since $h_\lambda = p_\lambda \circ h$ is an embedding, h is an embedding too. Hence, we can assume that $X = h(X) = Y$ is a subspace of $H(X)$ and $h(x) = x$ for each point $x \in X$.

Fix a continuous mapping $f : X \longrightarrow G \in \mathcal{V}$, where $f(p_X) = e \in G$. There exists $\eta \in M$ such that G_η is the submonoid of G generated by $f(X)$ and $f(x)$ $h_\eta(x)$ for each $x \in X$. Then $p_\eta(x) = \pi_\eta(h(x)) = f(x)$ for each $x \in X$. Since X generated $H(X)$, the homomorphism $\bar{f}$ is unique. Thus we can assume that $\pi_\eta = \bar{f}$ and $H(X)$ is the free topological monoid of the space X in the class $\mathcal{V}$. The existence of the free topological monoid of the space X is proved.

Let $F(X, \mathcal{V})$ and $F_1(X, \mathcal{V})$ be two free topological monoids of the space X. There existtwo continuous homomorphisms $h : F_1(X, \mathcal{V}) \longrightarrow F(X, \mathcal{V})$ and $g : F(X, \mathcal{V}) \longrightarrow F_1(X, \mathcal{V})$ such that $h(x) = g(x) = x$ for each $x \in X$. Consider the homomorphism $\varphi = h \circ g : F(X, \mathcal{V}) \longrightarrow F(X, \mathcal{V})$. That homomorphism is unique and is generated by the embedding of X in $F(X, \mathcal{V})$. Hence φ is the identical mapping and $h = g^{-1}$. Thus h and g are topological isomorphisms and the uniqueness of the free topological monoid of the space X is proved. □

Corollary 2.1.1. *Let $\mathcal{V}$ be a non-trivial quasivariety of topological monoids. Then for each space X there exists the unique abstract free monoid $F^a(X, \mathcal{V})$.*

Problem 2.1.1. *Let $\mathcal{V}$ be a non-trivial quasivariety of topological monoids. Under which conditions for a space X there exists the free topological monoid $F(X, \mathcal{V})$?*

Fix a space X for which there exists the free topological monoid $F(X, \mathcal{V})$. Then there exists a unique continuous homomorphism $\pi_X : F^a(X, \mathcal{V}) \longrightarrow F(X, \mathcal{V})$ such that $\pi_X(x) = x$ for each $x \in X$. The monoid $F(X, \mathcal{V})$ is called abstract free if π_X is a continuous isomorphism.

Problem 2.1.2. *Let $\mathcal{V}$ be a non-trivial quasivariety of topological monoids. Under which conditions for a space X there exists the free topological monoid $F(X, \mathcal{V})$, which is abstract free?*

The Problems 2.1.1 and 2.1.2 are important in the theory of universal algebras with topologies (see [133, 51, 52, 53, 54, 67]). These problems for varieties of topological algebras were posed by A. I. Maltsev ([133], see Maltsev's problems in section 1.6).

We say that a space X is zero-dimensional and denote $indX = 0$ if X has a base whose elements are open-and-closed [87].

Theorem 2.1.2. *Let $\mathcal{V}$ be a non-trivial quasivariety of topological monoids and there exists $H \in \mathcal{V}$ and point $b \in H$ such that $e \neq b$, and $E = \{e, b\}$ is a discrete subspace of H. Then for each zero-dimensional space X there exists the unique free topological monoid $F(X, \mathcal{V})$.*

Proof. Let $\{(U_\mu, V_\mu) : \mu M\}$ be a family of open-and-closed subsets of the space X with a fixed point p_X such that:

– $X = U_\mu \cup V_\mu$ and $U_\mu \cap V_\mu = \emptyset$ for each $\mu \in M$;

– if the set U is open in X, $x \in U$ and $x \neq p_X$, then there exists $\mu \in M$ such that $x \in V_\mu \subseteq U$;

– if the set U is open in X and $p_X \in U$, then there exists $\mu \in M$ such that $p_X \in U_\mu \subseteq U$.

We put $h_\mu(U_\mu) = \{e\}$ and $h_\mu(V_\mu) = \{b\}$. Then $h_\mu : X \longrightarrow H$ is a continuous mapping and the diagonal product $h : X \longrightarrow H^M$, where $h(x) = (h_\mu(x) : \mu \in M)$ for each point $x \in X$, is an embedding of X into $G = H^M$ and $h(p_X)$ is the unity of G. Theorem 2.1.1 completes the proof. □

The condition of the existence of a topological monoid H with a discrete space E is essential in the above theorem.

Example 2.1.1. *Let H be the topological monoid ω with the topology $\{\emptyset, H\} \cup \{U_n = \{i \in \omega : i \leq n\} : n \in \omega\}$. The set $\{0\}$ is open and dense in H. Let $\mathcal{V}(H)$ be the quasivariety of topological monoids generated by H. Any element of $\mathcal{V}(H)$ is a topological submonoid of the topological monoid H^M for some non-empty set M. In any $G \in \mathcal{V}(H)$ the unity $\{e\}$ is a dense subset. We have the following cases:*

Case 1. *If X is a space with the fixed point p_X and the set $\{p_X\}$ is closed in X (for instance, X is a T_1-space), then for X the free topological monoid $F(X, \mathcal{V}(H))$ does not exist.*

Case 2. *Let X be the space H with the fixed point $p_X = 0$. By virtue of Theorem 2.1.1, the free topological monoid $F(X, \mathcal{V}(H))$ of the space X exists.*

Case 3. *Let X be the space H with the fixed point $p_X \neq 0$. If $f : X \longrightarrow H$ is a continuous mapping and $f(p_X) = 0$ then $f(x) = 0$ for each $x \leq p_X$. Hence the free topological monoid $F(X, \mathcal{V}(H))$ of the space X =H with the fixed point $p_X \neq 0$ does not exist.*

2.2. Construction of the abstract free monoid

Fix a non-trivial quasivariety $\mathcal{V}$ of topological monoids. Consider a space X for which we can assume that $X \subseteq F^a(X, \mathcal{V})$ as a subset and $p_X = e$ is the unity (neutral element) in $F^a(X, \mathcal{V})$. In this case $e \in X \subseteq F^a(X, \mathcal{V})$. The set $A = X \setminus \{e\}$ is called an alphabet. If $n \geq 1$ and $x_1, x_2, ..., x_n \in X$, then the symbol $x_1x_2...x_n$ is called a word of the length n in the alphabet A. The word e is the empty word. Any word $x_1x_2...x_n$, where $x_1, x_2, ..., x_n \in X$, represents a unique element $x_1x_2...x_n = x_1 \cdot x_2 \cdot ... \cdot x_n \in F^a(X, \mathcal{V})$. A given element $b \in F^a(X, \mathcal{V})$ is represented by many words. There exists a word of the minimal length which represents the given element b. The length n of this word is called the length of the element b and we put $l(b) = n$. If the element b is represented by the words $x_1x_2...x_n$, $y_1y_2...y_m$ of the minimal length, then $n = m$ and $\{x_1, x_2, ..., x_n\} = \{y_1, y_2, ..., y_m\}$. In this case we say that the word $x_1x_2...x_n$ is irreducible and that $Sup(b) = \{x_1, x_2, ..., x_n\}$ is the support of the element b. If the element b is represented by the words $x_1x_2...x_n$, $y_1y_2...y_n$ of the minimal length, then there exists a bijection $h : \{1, 2, ..., n\} \longrightarrow \{1, 2, ..., n\}$ such that $x_i = y_{h(i)}$ for each $i \leq n$. Obviously, $Sup(e) = \{e\}$ and $e \notin Sup(b)$ if $b \neq e$. If $e \in Y \subseteq X$, $b \in F^a(X, \mathcal{V})$ and $Sup(b) \subseteq Y$, then $b \in F^a(Y, \mathcal{V})$. In particular, $F^a(Y, \mathcal{V})$ is the submonoid of $F^a(X, \mathcal{V})$ generated by the set Y.

For any two elements $a, b \in F^a(Y, \mathcal{V})$ we put $Sup(a, b) = Sup(a) \cup Sup(b) \cup \{e\}$. In particular, $Sup(a, a) = Sup(a) \cup \{e\}$.

Remark 2.2.1. *Let $b \in F^a(X, \mathcal{V})$ and $b \neq e$. Then $x \in Sup(b)$ if and only if $x \neq e$ and $b \notin F^a(X \setminus \{x\}, \mathcal{V})$.*

Remark 2.2.2. *Let $b = x_1x_2...x_n \in F^a(X, \mathcal{V})$. Then we have $Sup(b) \subseteq Sup(b, b) \subseteq \{e, x_1, x_2, ..., x_n\}$.*

Remark 2.2.3. *If $\mathcal{V}$ is the variety of all topological monoids, then any $b \in F^a(X, \mathcal{V})$ is represented by some word of the minimal length. If the monoids from $\mathcal{V}$ are commutative and p_X, a, b are distinct elements of X, then ab and ba are distinct words, but $ab = ba$ in $F^a(Y, \mathcal{V})$.*

2.3. On the non-Burnside quasivarieties

A quasivariety $\mathcal{V}$ of topological monoids is called a Burnside quasivariety if there exist two minimal numbers $p = p(\mathcal{V}), q = q(\mathcal{V}) \in \omega$ such that $0 \leq q < p$ and $x^p = x^q$ for all $x, y \in G \in \mathcal{V}$. In this case any $G \in \mathcal{V}$ is a (p, q)-periodic monoid of the exponent (p, q). If $q = 0$, then any monoid $G \in \mathcal{V}$ is a periodic monoid of the exponent p and $x^p = e$ for each $x \in G \in \mathcal{V}$.

The trivial quasivariety is considered Burnside of the exponent $(0, 1)$.

Example 2.3.1. *Fix $0 \leq q < p$ and an element $b \neq e$. We put $b^0 = e$, $b^1 = b$ and $b^{n+1} = b^n \cdot b = b \cdot b^n$ for each $n \in \mathbb{N}$. We consider that $b^p = b^q$ and all elements $\{b^i : i < p\}$ are distinct. Then $G_{(p,q)} = \{b^n : n \in \mathbb{N}\} = \{b^i : i < p\}$ is a monoid and $|G_{(p,q)}| = p$. Denote by $\mathcal{W}_{(p,q)}$ the complete variety of topological monoids generated by the discrete monoid $G_{(p,q)}$, i.e. is the minimal class of topological monoids with the properties:*

– the class $\mathcal{W}_{(p,q)}$ is a complete quasivariety of topological monoids;

– $G_{(p,q)} \in \mathcal{W}_{(p,q)}$

– if $f : A \to B$ is a continuous homomorphism of a topological monoid A onto a a topological monoid B, $A \in \mathcal{W}_{(p,q)}$ and B is a T_0-space, then $B \in \mathcal{W}_{(p,q)}$.

Then $\mathcal{W}_{(p,q)}$ is a variety of topological commutative monoids of the exponent (p, q).

Example 2.3.2. *Let $\mathcal{W}_\omega$ is the complete quasivariety generated by the discrete monoid $\omega = \{0, 1, 2, \ldots\}$ with the additive operation. The class $\mathcal{W}_\omega$ is a non-Burnside quasivariety of commutative topological monoids.*

Theorem 2.3.1. *Let $\mathcal{V}$ be a non-trivial Burnside quasivariety of the exponent $p \geq 2$. Then:*

1. Each topological monoid $G \in \mathcal{V}$ is a topological group.

2. If d is a stable pseudo-quasi-metric on $G \in \mathcal{V}$, then d is a pseudo-metric on G and $d(x, y) = d(y, x) = d(xz, yz) = d(zx, zy) = d(y^{-1}, x^{-1}) \leq (p - 1)d(y, x)$ for all $x, y, z, \in G \in \mathcal{V}$.

3. If $p = 2$ and d is a stable pseudo-quasi-metric on $G \in \mathcal{V}$, then d is a pseudo-metric on G.

Proof. Let $x \in G \in \mathcal{V}$ and $p(x) = min\{q \in \mathbb{N} : x^q = e\}$. If $p(x) \geq 2$, then $x^{p(x)} = e$. Thus we can assume that $x^{p(x)-1} = x^{-1}$. Thus G is a group. If d is a stable pseudo-quasi-metric on G, then $d(x, y) = d(xz, yz) = d(zx, zy) = d(y^{-1}xx^{-1}, y^{-1}yx^{-1}) = d(y^{-1}, x^{-1})$ for all $x, y, z, \in G$. If $p = 2$, then $x = x^{-1}$. Assertion 2 is proved. Assertion 3 follows from Assertion 2.

Let $G \in \mathcal{V}$ be a paratopological group. A topological group is a paratopological group with a continuous inverse operation $x \to x^{-1}$. Since the inverse operation $x \to x^{p-1} = x^{-1}$ is continuous, Assertion 1 is proved. The proof is complete. □

Theorem 2.3.2. *Let $\mathcal{V}$ be a non-trivial quasivariety of topological monoids. Then the following assertions are equivalent:*

1. $\mathcal{V}$ is a non-Burnside quasivariety.

2. On ω there exists a topology T for which $(\omega, T) \in \mathcal{V}$.

Proof. Implication $2 \to 1$ is obvious. Assume that $\mathcal{V}$ is a non-Burnside quasivariety. Let $\{(p_n, q_n) : n \in \mathbb{N}\}$ is the collection of all pairs $(p, q) \in \omega \times \omega$ such that $q < p$. For each

$n \in \mathbb{N}$ there exist $G_n \in \mathcal{V}$ and $a_n \in G_n$ such that all elements $a_n^0 = e, a_n^1, a_n^2, ..., a_n^{p_n-1}$ are distinct and $a_n^{p_n} = a_n^{q_n}$. We put $G = \Pi\{G_n : n \in \mathbb{N}\}$ and $a = (a_n : n \in \mathbb{N})$. Then $a \in G \in \mathcal{V}$. We put $H = \{a^n : n \in \omega\}$. Then $H \in \mathcal{V}$ is a submonoid of the monoid G. The mapping $n \to a^n$ is a isomorphism of ω onto H. Implication $1 \to 2$ and the theorem are proved. □

Corollary 2.3.1. *Let $\mathcal{V}$ be a non-Burnside quasivariety, X be a space, $b = x_1x_2...x_n \in F^a(X, \mathcal{V})$, $l(b) = m$ and $Sup(b) = \{y_1, y_2, ..., y_s\}$. Then:*

1. If $b = e$, then $s = 1$, $m = 0$ and $x_i = y_1 = e$ for each $i \leq n$.

2. Let $b \neq e$. Then $n \geq m \geq s \geq 1$ and $\{y_1, y_2, ..., y_s\} \subseteq \{x_1, x_2, ..., x_n\} \subseteq \{e\} \cup \{y_1, y_2, ..., y_s\}$, i.e. for each $i \leq n$ we have $x_i \in Sup(b, b)$. Moreover, if $A = \{i \leq n : x_i \neq e\}$, then there exists a mapping $h : A \longrightarrow \{1, 2, ..., s\}$ such that $h(A) = \{1, 2, ..., m\}$, $A = \{i_1, i_2, ..., i_m\}$, $x_i = y_{h(i)}$ for each $i \in A$ and $x = [x_{i_1}x_{i_2}...x_{i_m}]$ is an irreducible word.

3. $Sup(b) \subseteq \{x_1, x_2, ..., x_n\} \subseteq Sup(b, b)$.

Corollary 2.3.2. *Let $\mathcal{V}$ be a non-Burnside quasivariety, X be a space and $b = x_1x_2...x_m = y_1, y_2, ..., y_m \in F^a(X, \mathcal{V})$ and $x_i \neq e$ for each $i \leq m$. Then there exists a one-to-one mapping $h : \{1, 2, ..., m\} \longrightarrow \{1, 2, ..., m\}$ such that $x_i = y_{h(i)}$ for each $i \leq m$.*

Remark 2.3.1. *Assertions of Corollary 2.3.1 are not true for Burnside quasivarieties. Consider the quasivariety $\mathcal{W}_{(0,2)}$ of topological monoids (groups) with the identity $x^2 = e$. Let $X = \{e, a, b, c\}$ be a discrete space with four distinct points. Then $z = a = cabeeaecba = bba = acc \in F^a(X, \mathcal{W}_{(0,2)})$ and $Sup(z) = \{a\}$.*

The following theorem solves Problem 2.1.1 for complete non-Burnside quasivarieties of topological monoids.

Theorem 2.3.3. *Let $\mathcal{V}$ be a complete non-Burnside quasivariety of topological monoids. Then for each T_0-space X there exists the free topological monoid $F(X, \mathcal{V})$.*

Proof. By virtue of Theorem 2.3.2 the discrete monoid ω is an element of $\mathcal{V}$. Denote by ω_l the monoid ω with the topology $T_l = \{\emptyset, \omega\} \cup \{V_n = \{i \in \omega : i \leq n\} : n \in \omega\}$ and by ω_r the monoid ω with the topology $T_r = \{\emptyset, \omega\} \cup \{W_n = \{i \in \omega : i \geq n\} : n \in \omega\}$. Obviously, the topological monoids ω_l and ω_r are elements of $\mathcal{V}$.

Consider a space X with the fixed point p_X. Let U be an open subset of the space X. We construct a topological monoid $G_U \in \mathcal{V}$ with the unity e_U and a continuous mapping $h_U : X \longrightarrow G_U$ such that $h_U(p_X) = e_U$ and $U = h_U^{-1}(h_U(U))$. For that we consider two cases.

Case 1. $p_X \in U$.

In this case we put $G_U = \omega_l$, $h_U(U) = \{0\}$ and $h_U(X \setminus U) = \{1\}$.

Case 2. $p_X \notin U$.

In this case we put $G_U = \omega_r$, $h_U(U) = \{1\}$ and $h_U(X \setminus U) = \{0\}$.

Now consider the diagonal product $h : X \longrightarrow G = \Pi\{G_U : U \textit{ is open subset of } X\}$, where $h(x) = (h_U(x) : U \textit{ is open subset of } X)$ for each $x \in X$. By construction, $G \in \mathcal{V}$, h is an embedding of X in G and $h(p_X) = e$ is the neutral element in G. Theorem 2.1.1 completes the proof. □

The following theorem solves Problem 2.1.1 for complete non-trivial quasivarieties of topological monoids.

Theorem 2.3.4. *Let $\mathcal{V}$ be a complete non-trivial quasivariety of topological monoids. Then for each completely regular space X there exists the free topological monoid $F(X, \mathcal{V})$.*

Proof. In [51] it was proved that any topological monoid $G \in \mathcal{V}$ is a submonoid of some arcwise connected topological monoid from $\mathcal{V}$. Hence there exists a topological monoid $H \in \mathcal{V}$ such that the closed interval $[0, 1]$ is a subspace of H and $e = 0$ is the neutral element in H.

Let βX be the Stone-Čech compactification of the given completely regular space with the fixed point p_X. Let $\{(U_\mu, F_\mu) : \mu \in M\}$ be the collection of all pairs (U, F), where U is an open subset of the space βX, F is a closed subset of the space βX and $F \subseteq U$ and $p_X \in F$ provided $p_X \in U$. We construct a topological monoid $G_\mu = H \in \mathcal{V}$ with the unity e_μ and a continuous mapping $h_\mu : X \longrightarrow G_\mu$ such that $h_\mu(p_X) = e_\mu$ and $h_\mu(F_\mu) \cap h_\mu(X \setminus U_\mu) = \emptyset$. For that we consider two cases.

Case 1. $p_X \in U_\mu$.

In this case we fix a continuous mapping $h : X \longrightarrow [0, 1] \subseteq H = G_\mu$ such that $h_\mu(F_\mu) = \{0\}$ and $h_\mu(X \setminus U_\mu) = \{1\}$.

Case 2. $p_X \notin U_\mu$.

In this case we fix a continuous mapping $h : X \longrightarrow [0, 1] \subseteq H = G_\mu$ such that $h_\mu(F_\mu) = \{1\}$ and $h_\mu(X \setminus U_\mu) = \{0\}$.

Now consider the diagonal product $h : X \longrightarrow G = \Pi\{G_\mu : \mu \in M\}$, where $h(x) = (h_\mu(x) : \mu \in M)$ for each $x \in X$. By construction, $G \in \mathcal{V}$, h is an embedding of X in G and $h(p_X) = e$ is the neutral element in G. Theorem 2.1.1 completes the proof. □

The following corollary follows from Theorems 2.3.1 and 2.3.3.

Corollary 2.3.3. *Let $\mathcal{V}$ be a complete non-trivial Burnside quasivariety of the exponent $p \geq 2$. Then for a space X there exists the free monoid $F(X, \mathcal{V})$ if and only if the space X is Tychonoff.*

Completeness of quasivariety $\mathcal{V}$ is essential in the conditions of the above two theorems.

Example 2.3.3. *Let H be a discrete monoid and $\mathcal{V}(H)$ the quasivariety of topological monoids generated by H. Any element of $\mathcal{V}(H)$ is a topological submonoid of the topological monoid H^M for some non-empty set M. Hence, for a space X there exists the free monoid $F(X, \mathcal{V})$ if and only if the space X is Tychonoff and $indX = 0$.*

Example 2.3.4. *Let ω_r be the monoid ω with the topology $T_r = \{\emptyset, \omega\} \cup \{W_n = \{i \geq n : n \in \omega\}\}$ and $\mathcal{V}(\omega_r)$ be the quasivariety of topological monoids generated by ω_r. Any element of $\mathcal{V}(\omega_r)$ is a topological submonoid of the topological monoid ω_r^M for some non-empty set M. For a space X there exists the free monoid $F(X, \mathcal{V})$ if and only if the space X is a T_0-space and the set $\{p_X\}$ is closed in X. Denote by Z an infinite space with a fixed point p_Z and the topology $\{\emptyset, Z\} \cup \{U \subseteq Z : p_Z \in U\}$. The subset $\{p_Z\}$ is open and dense in Z. Moreover, if $f : Z \longrightarrow \omega_r$ is a continuous mapping and $f(p_Z) = 0$, then $f(Z) = \{0\}$. Thus the free topological monoid for the space Z in the quasivariety $\mathcal{V}(\omega_r)$ does not exist.*

2.4. Extension of pseudo-quasi-metrics

Lemma 2.4.1. *Let d_1, d_2 be two pseudo-quasi-metrics on a monoid G. Then:*

1. $d(x, y) = sup\{d_1(x, y), d_2(x, y)\}$ is a pseudo-quasi-metric on G.

2. If the pseudo-quasi-metrics d_1, d_2 are invariant on G, then the pseudo-quasi-metric d is invariant on G too.

Proof. Fix $x, y, z, v \in G$. Then $d(x, z) = sup\{d_1(x, z), d_2(x, z)\} \leq sup\{d_1(x, y) + d_1(y, z), d_2(x, y) + d_2(y, z)\} \leq sup\{d_1(x, y), d_2(x, y)\} + sup\{d_1(y, z), d_2(y, z)\} = d(x, y) + d(y, z)$. Hence d is a pseudo-quasi-metric on G.

Assume that the pseudo-quasi-metrics d_1, d_2 are invariant on G. We observe that $d(zxv, zyv) = sup\{d_1(zxv, zyv), d_2(zxv, zyv)\} \leq sup\{d_1(x, y), d_2(x, y)\} = d(x, y)$. Thus the pseudo-quasi-metric d is invariant too. □

Fix a non-trivial complete quasivariety $\mathcal{V}$ of topological monoids. Consider a non-empty set X with a fixed point $e \in X$. We assume that $e \in X \subseteq F^a(X, \mathcal{V})$ and e is the identity of the monoid $F^a(X, \mathcal{V})$. Let ρ be a pseudo-quasi-metric on the set X. Denote by $Q(\rho)$ the set of all stable pseudo-quasi-metrics d on $F^a(X, \mathcal{V})$ for which $d(x, y) \leq \rho(x, y)$ for all $x, y \in X$. The set $Q(\rho)$ is non-empty, since it contains the trivial pseudo-quasi-metric $d(x, y) = 0$ for all $x, y \in F^a(X, \mathcal{V})$. For all $a, b \in F^a(X, \mathcal{V})$ we put $\hat{\rho}(a, b) = sup\{d(a, b) : d \in Q(\rho)\}$. We say that $\hat{\rho}$ is the maximal stable extension of ρ on $F^a(X, \mathcal{V})$.

Property 2.4.1. $\hat{\rho} \in Q(\rho)$.

Proof. Obviously $d(x, y) \leq \rho(x, y)$ for $x, y \in X$. Let $d \in Q(\rho)$. Fix two points $a, b \in F^a(X, \mathcal{V})$. There exists $n \in \mathbb{N}$ and $x_1, y_1, x_2, y_2, ..., x_n, y_n \in X$ such that $a = x_1x_2...x_n$ and $b = y_1y_2...y_n$. Then $d(a, b) \leq \Sigma\{d(x_i, y_i) : i \leq n\} \leq \Sigma\{\rho(x_i, y_i) : i \leq n\}$. Hence $\rho(a, b) \leq sup\{\Sigma\{d(x_i, y_i) : i \leq n\} : d \in Q(\rho)\} \leq \Sigma\{\rho(x_i, y_i) : i \leq n\} < +\infty$. Therefore, by virtue of Lemma 2.4.1, $\hat{\rho}$ is a stable pseudo-quasi-metric from the set $Q(\rho)$. □

For any $r > 0$ we put $d_r(a, a) = 0$ and $d_r(a, b) = r$ for all distinct points $a, b \in F^a(X, \mathcal{V})$. Then d_r is an invariant metric on $F^a(X, \mathcal{V})$.

Property 2.4.2. *Let $r > 0$ and $\rho(x, y) \geq r$ for all distinct points $x, y \in X$. Then $\hat{\rho}$ is a quasi-metric on $F^a(X, \mathcal{V})$, $d_r \in Q(\rho)$ and $\hat{\rho}(a, b) \geq r$ for all distinct points $a, b \in F^a(X, \mathcal{V})$.*

Proof. IF we put $d(x, y) = r$ for $x \neq y$ and $d(x, x) = 0$, then d is a metric on $F^a(X, \mathcal{V})$ and $d \in Q(\rho)$. In this case $\hat{\rho} \geq d$. The proof is complete. □

For any $a, b \in F^a(X, \mathcal{V})$ we put $\bar{\rho} = inf\{\Sigma\{\rho(x_i, y_i) : i \leq n\} : n \in \mathbb{N}, x_1, y_1, x_2, y_2, ..., x_n, y_n \in X, a = x_1x_2...x_n, b = y_1y_2...y_n\}$ and $\rho^*(a, b) = inf\{\bar{\rho}(a, z_1) +... + \bar{\rho}(z_i, z_{i+1}) + ... + \bar{\rho}(z_n, b) : n \in \mathbb{N}, z_1, z_2, ..., z_n \in F^a(X, \mathcal{V})\}$.

Property 2.4.3. *$\bar{\rho}$ is a pseudo-distance on $F^a(X, \mathcal{V})$ and $\bar{\rho}(x, y) \leq \rho(x, y)$ for all $x, y \in X$.*

Proof. Obviously, $\bar{\rho}$ is a pseudo-distance. If $a, b \in X$, then $a = ae = a$, $b = be = b$ and $\bar{\rho}(a, b)$ $= inf\{\Sigma\{\rho(x_i, y_i) : i \leq n\} : n \in \mathbb{N}, x_1, y_1, x_2, y_2, ..., x_n, y_n \in X, a = x_1x_2...x_n, b = y_1y_2...y_n\}$ $\leq \rho(a, b)$. □

Property 2.4.4. *Let $\mathcal{V}$ be a non-Burnside quasivariety. Then $\bar{\rho}(x, y) = \rho(x, y)$ for all $x, y \in X$.*

Proof. Assume that $n \in \mathbb{N}$, $x_1, y_1, x_2, y_2, ..., x_n, y_n \in X$, $x = x_1x_2...x_n$ and $y = y_1y_2...y_n$. There exist $i, j \leq n$ for which $x = x_i$ and $y = y_j$. We have two possible cases.

Case 1. $i = j$.

In this case, as was mention in Corollary 2.3.1, $x_k = y_k = e$ for each $k \neq i$. Thus $\Sigma\{\rho(x_i, y_i) : i \leq n\} = \rho(x_i, y_i) = \rho(x, y)$.

Case 2. $i \neq j$.

In this case, as was mention in Corollary 2.3.1, we have $x_j = y_i = e$. Hence $\Sigma\{\rho(x_i, y_i) : i \leq n\}$ $\geq \rho(x_i, y_i) + \rho(x_j, y_j) = \rho(x, e) + \rho(e, y) \geq \rho(x, y)$. The proof is complete. □

Property 2.4.5. *The pseudo-distance $\bar{\rho}$ is stable on $F^a(X, \mathcal{V})$.*

Proof. Fix $a, b, c \in F^a(X, \mathcal{V})$ and $\varepsilon > 0$. Let $c = z_1z_2...z_m$. There exist $n \in \mathbb{N}$ and the words $a = x_1x_2...x_n$, $b = y_1y_2...y_n$ such that $\bar{\rho}(a, b) \leq \Sigma\{\rho(x_i, y_i) : i \leq n\} < \rho(a, b) + \varepsilon$. Then $\bar{\rho}(ac, bc) = \bar{\rho}(x_1x_2...x_nz_1z_2...z_m, y_1y_2...y_nz_1z_2...z_m) \leq \Sigma\{\rho(x_i, y_i) : i \leq n\} < \bar{\rho}(a, b) + \varepsilon$. Hence $\bar{\rho}(ac, bc) \leq \bar{\rho}(a, b)$. The proof of inequality $\bar{\rho}(ca, cb) \leq \bar{\rho}(a, b)$ is similar. Proposition 1 proved in [56] about the equivalence of the properties of invariante and stability of a pseudo-quasi-metric on a semigroup completes the proof. □

Property 2.4.6. *The pseudo-distance ρ^* is a stable pseudo-quasi-metric on $F^a(X, \mathcal{V})$ and $\rho^* \in Q(\rho)$.*

Proof. Follows from Properties 2.4.2 and 2.4.4. □

In the following properties we assume that $\mathcal{V}$ is a non-Burnside quasivariety.

Property 2.4.7. *If ρ is a quasi-metric on X, then $\bar{\rho}$ is a distance on $F^a(X, \mathcal{V})$.*

Proof. Assume that ρ is a quasi-metric on X and $\bar{\rho}$ is not a distance on $F^a(X, \mathcal{V})$. There exist two distinct points $b, c \in F^a(X, \mathcal{V})$ such that $\bar{\rho}(b, c) = \bar{\rho}(c, b) = 0$. Suppose that $n \geq 2$ and $l(b) + l(c) \leq n$. Then $\bar{\rho}(b, c) = inf\{\Sigma\{\rho(x_i, y_i) : i \leq m\} : m \in \mathbb{N}, m \leq 4n^2, x_1, x_2, ..., x_m \in Sup(b, b), y_1, y_2, ..., y_m \in Sup(c, c), b = x_1x_2...x_m, c = [y_1y_2...y_m]\}$.

Since $\bar{\rho}(b, c) = 0$, there exist $m \in \mathbb{N}$, $x_1, x_2, ..., x_m \in Sup(\{b\}) \cup \{e\}$, and $y_1, y_2, ..., y_m \in Sup(\{c\}) \cup \{e\}$ such that $b = x_1x_2...x_m$, $c = y_1y_2...y_m$ and $\bar{\rho}(b, c) = \Sigma\{\rho(x_i, y_i) : i \leq m\} = 0$. Since $\bar{\rho}(c, b) = 0$, there exist $k \in \mathbb{N}$, $c_1, c_2, ..., c_k \in Sup(\{c\}) \cup \{e\}$, $b_1, b_2, ..., b_k \in Sup(\{b\}) \cup \{e\}$ such that $b = b_1b_2...b_k$, $c = c_1c_2...c_k$ and $\bar{\rho}(c, b) = \Sigma\{\rho(c_j, b_j) : j \leq k\} = 0$.

Fix $i_1 \leq m$. Then $\rho(x_{i_1}, y_{i_1}) = 0$. There exists j_1 such that $c_{j_1} = y_{i_1}$. Then $\rho(c_{j_1}, b_{j_1}) = 0$. There exists i_2 such that $x_{i_2} = b_{j_1}$. Then $\rho(x_{i_2}, y_{i_2}) = 0$ and so on. As a result, we obtain a sequence x_{i_1}, $y_{i_1} = c_{j_1}$, $b_{j_1} = x_{i_2}$, $y_{i_2} = c_{j_2}$, ..., x_{i_p}, $y_{i_p} = c_{j_p}$, $b_{j_p} = x_{i_{p+1}}$, $y_{i_{p+1}} = c_{j_{p+1}}$, ... such that $\rho(x_{i_p}, y_{i_p})$ $= \rho(c_{j_p}, b_{j_p}) = 0$ for any $p \in \mathbb{N}$. Since $x_{i_1}, x_{i_2}, \ldots, x_{i_p}, \ldots$ are elements of a finite set $Sup(b, b) = Sup(b) \cup \{e\}$, there exist two numbers $p, q \in \mathbb{N}$ such that $q < p$ and $x_{i_q} = x_{i_p}$. Hence $\rho(x_{i_q}, y_{i_q}) = 0$ and $0 \leq \rho(y_{i_q}, x_{i_q}) = \rho(y_{i_q}, x_{i_p}) \leq \rho(y_{i_q}, c_{j_q}) + \rho(c_{j_q}, b_{j_q}) + \rho(x_{i_{q+}}, y_{i_{q+1}}) + \ldots + \rho(c_{j_{p-1}}, b_{p_{p-1}}) + \rho(b_{j_{p-1}}, x_{i_p}) = 0$, a contradiction. The proof is complete. □

Property 2.4.7 is not true for Burnside quasivarieties.

Example 2.4.1. *Let $n \in \mathbb{N}$ and $n \geq 2$. Consider the quasivariety $\mathcal{W}$ of topological monoids (groups) with the identities $x^n = e$. Let $\prec$ be a linear ordering on a set X, $|X| \geq 2$, and $e \leq x$ for each $x \in X$. We put $\rho(x, x) = 0$ for each $x \in X$ and for distinct $x, y \in X$ with $x \prec y$ we put $\rho(x, y)$ $= 1$ and $\rho(y, x) = 0$. Then ρ is a quasi-metric on X. Fix $a, b \in X$ with $a \leq b$. Then $\bar{\rho}(b, a) = 0$ and $\bar{\rho}(a, b) = \bar{\rho}(b^n a, be^n) \leq \rho(b, a) + (n-1)\rho(b, e) + \rho(a, e) = 0$.*

Fix now $a, b \in F^a(X, \mathcal{W})$. There exists $m \in \mathbb{N}$ and $x_1, y_1, x_2, y_2, ..., x_m, y_m \in X$ such that $a = x_1x_2...x_m$ and $b = y_1y_2...y_m$. By virtue of Property 2.4.5, we have $0 \leq \bar{\rho}(a, b) = \bar{\rho}(x_1x_2...x_m, y_1y_2...y_m) \leq \Sigma\{\bar{\rho}(x_i, y_i) : i \leq m\} = 0$. Hence $\bar{\rho}(x, y) = 0$ for all $x, y \in F^a(X, \mathcal{W})$. Therefore $\bar{\rho}(x, y) = 0$ for all $x, y \in F^a(X, \mathcal{W})$.

Example 2.4.2. *Let $p, q \in \mathbb{N}$ and $1 \leq q < p = q + k$. Consider the non-trivial quasivariety $\mathcal{W}$ of topological monoids with the identity $x^q = x^p$. Fix a set X with three distinct elements $\{e, a, b\}$. Let $\prec$ be a linear ordering on a set X and $e \prec a \prec b$. We put $\rho(x, x) = 0$ for each $x \in X$ and for distinct $x, y \in X$ with $x \prec y$ we put $\rho(x, y) = 1$ and $\rho(y, x) = 0$. Then ρ is a quasi-metric on X. We have $\rho(x, x) = 0$ for each $x \in X$, $\rho(e, a) = \rho(e, b) = \rho(a, b) = 1$ and $\rho(b, a) = \rho(a, e) = \rho(b, e) = 0$.*

We put $u = b^q \in F^a(X, \mathcal{W})$ and $v = a^q b^q \in F^a(X, \mathcal{W})$. There exist two numbers $k, m \in \mathbb{N}$ for which $q + k(p - q) = 2q + m$. By construction, $\hat{\rho}(v, u) = \hat{\rho}(a^q b^q, e^q b^q) \leq q(\rho(a, e) + \rho(b, b))$

$= 0$ *and* $\hat{\rho}(u, v) = \bar{\rho}(b^q, a^q b^q) = \bar{\rho}(b^{q+k(p-q)}, a^q b^q e^m) = \bar{\rho}(b^q b^q b^m, a^q b^q e^m) = q\rho(b, a) + q\rho(b, b) + m\rho(b, e) = 0$. *Hence* $\hat{\rho}(x, y) + \hat{\rho}(v, u) = 0$. *Therefore* $\bar{\rho}(u, v) + \bar{\rho}(v, u) = 0$.

Example 2.4.3. *Consider the quasivariety* $\mathcal{V} = \mathcal{W}_{(0,2)}$ *of topological monoids with the identity* $x^2 = e$. *Let* $X = \{e, a, b\}$, $\rho(x, x) = 0$ *for each* $x \in X$, $\rho(a, b) = \rho(e, a) = \rho(b, e) = 0$, $\rho(b, a) = \rho(a, e) = \rho(e, b) = 1$. *We have* $F^a(X, \mathcal{V}) = \{e, a, b, ab\}$ *and* $ab = ba$. *In this case* ρ *is not a quasi-metric and* $\bar{\rho}(b, a) = \bar{\rho}(be, ea) = 0 < \rho(b, a) = 1$, $\bar{\rho}(a, b) = \rho(a, b) = 0$, $\bar{\rho}(a, ab) = \bar{\rho}(ea, bb) = 0$, $\bar{\rho}(ab, a) =\bar{\rho}(ab, ae) = 0$, $\bar{\rho}(ab, b) = \bar{\rho}(ab, be) = 0$, $\bar{\rho}(b, ab) = \bar{\rho}(eb, ab) = 0$, $\bar{\rho}(e, b) = \bar{\rho}(bb, be) = 0$, $\bar{\rho}(ab, e) = \bar{\rho}(ab, bb) = 0$, $\bar{\rho}(e, ab) = \bar{\rho}(ebb, aeb) = 0$, $\bar{\rho}(e, b) = \bar{\rho}(ebb, eeb) = 0$. *Hence* $\bar{\rho} = \hat{\rho}$ *is the trivial pseudo-metric on* $F^a(X, \mathcal{V})$.

Property 2.4.7 is not true for distances which are not quasi-metrics.

Example 2.4.4. *Consider a non-trivial quasivariety* $\mathcal{V}$ *of topological monoids. Let* $X = \{e, a, b\}$, $\rho(x, x) = 0$ *for each* $x \in X$, $\rho(a, b) = \rho(e, a) = \rho(b, e) = 0$, $\rho(b, a) = \rho(a, e) = \rho(e, b) = 1$. *In this case* $\bar{\rho}(b, a) = \bar{\rho}(be, ea) = 0 < \rho(b, a) = 1$ *and* $\bar{\rho}(a, b) = \rho(a, b) = 0$.

Property 2.4.8. *Let* $a, b \in F^a(X, \mathcal{V})$ *be two distinct points in* $F^a(X, \mathcal{V})$ *and* $r(a, b) = min\{\rho(x, y) : x \in Sup(a, a), y \in Sup(b, b), x \neq y\}$. *Then* $\hat{\rho}(a, b) = \rho^*(a, b) \geq r(a, b)$.

Proof. Assume that $r(a, b) - \rho^*(a, b) = 3\delta > 0$. There exist $n \in \mathbb{N}$ and $z_1, z_2, ..., z_n \in F^a(X, \mathcal{V})$ such that $\rho^*(a, b) \leq \bar{\rho}(a, z_1)$ +...+ $\bar{\rho}(z_i, z_{i+1})$ + ... + $\bar{\rho}(z_n, b) < \rho^*(a, b) + \delta$. Let $z_0 = a$ and $z_{n+1} = b$. For each $i \in \{0, 1, 2, ..., n\}$ there exist the representations $z_i = u_{(i,1)}u_{(i,2)}...u_{(i,m_i)}$ and $z_{i+1} = v_{(i,1)}v_{(i,2)}...v_{(i,m_i)}$ such that $\{u_{(i,1)}, u_{(i,2)}, ..., u_{(i,m_i)}\} \subseteq Sup(z_i, z_i)$, $\{v_{(i,1)}, v_{(i,2)}, ..., v_{(i,m_i)}\} \subseteq Sup(z_{i+1}, z_{i+1})$ and $\bar{\rho}(z_i, z_{i+1}) \leq \Sigma\{\rho(u_{(i,j)}, v_{(i,j)} : j \leq m_i\} \leq \bar{\rho}(z_i, z_{i+1}) \leq \delta/(n + 1)$. Without lost of generality, we can assume that there exists $m \in \mathbb{N}$ such that $m_i = m$ for each $i \in \{0, 1, 2, ..., n\}$. For each $i \in \{0, 1, 2, ..., n\}$ there exists a one-to-one mapping $h_i : \{1, 2, ..., m\} \longrightarrow \{1, 2, ..., m\}$ such that $v_{(i,j)} = u_{(i+1,h_i(j))}$ for each $j \leq m$. Then the chain $j_0 = j$, $j_1 = h_1(j)$, $j_2 = h_2(j_1)$, ..., $j_n = h_n(j_{n-1})$ and the number $r_j = \rho(u_{(0,j_0)}, v_{(0,j_0)}) + \rho(u_{(1,j_1)}, v_{(1,j_1)})$ + ...+ $\rho(u_{(n,j_n)}, v_{(n,j_n)}) \geq \rho(u_{(0,j_0)}, v_{(n,j_n)})$ are determined for any $j \leq m$. We put $h(j) = j_n$. Then $h : \{1, 2, ..., m\} \longrightarrow \{1, 2, ..., m\}$ is a one-to-one mapping as the composition of the mappings $h_1, h_2, ..., h_n$. We obtain that $\rho^*(a, b) + 3\delta \leq \bar{\rho}(a, z_1), ..., \bar{\rho}(z_i, z_{i+1}$ + ... + $\bar{\rho}(z_n, b) \geq \bar{\rho}(a, b)\ r(a, b)$. The proof is complete. □

The following properties follow from Property 2.4.8.

Property 2.4.9. *If* ρ *is a quasi-metric on* X, *then* ρ^* *and* $\hat{\rho}$ *are quasi-metrics on* $F^a(X, \mathcal{V})$.

Property 2.4.10. *If* ρ *is a strong quasi-metric on* X, *then* ρ^* *and* $\hat{\rho}$ *are strong quasi-metrics on* $F^a(X, \mathcal{V})$.

Proved properties lead us to the following general result:

Theorem 2.4.1. *Let ρ be a pseudo-quasi-metric on X, Y be a subspace of X and $e \in Y$. Denote by $M(Y) = F^a(Y, \mathcal{V})$ the submonoid of the monoid $F^a(X, \mathcal{V})$ generated by the set Y and by d_Y the extension of $\hat{\rho}|Y$ on $M(Y)$ of the pseudo-quasi-metric ρ_Y on Y, where $\rho_Y(y, z) = \rho(y, z)$ for all $y, z \in Y$. Then:*

1. *$d_Y(a, b) = \hat{\rho}(a, b)$ for all $a, b \in M(Y)$.*

2. *If $\mathcal{V}$ is a non-Burnside quasivariety, then $\bar{\rho}(x, y) = \rho(x, y)$ for all $x, y \in X$.*

3. *If ρ is a (strong) quasi-metric on Y, then $\hat{\rho}$ is a (strong) quasi-metric on $M(Y)$.*

4. *If ρ is a metric on Y, then $\hat{\rho}$ is a metric on $M(Y)$.*

5. *If $a, b \in F^a(Y, \mathcal{V})$ are distinct points and ρ is a quasi-metric on $Sup(a, b)$, then $\hat{\rho}(a, b) + \hat{\rho}(b, a) > 0$.*

6. *If $a, b \in F^a(Y, \mathcal{V})$ are distinct points and ρ is a strong quasi-metric on $Sup(a, b)$, then $\hat{\rho}(a, b) > 0$ and $\hat{\rho}(b, a) > 0$.*

7. *For any $a, b \in F^a(Y, \mathcal{V})$ there exist $n \in \mathbb{N}$, $x_1, x_2, ..., x_n \in Sup(a, a)$ and $y_1, y_2, ..., y_n \in Sup(b, b)$ such that $a = x_1x_2...x_n$, $b = y_1y_2...y_n$, $n \leq l(a) + l(b)$ and $\bar{\rho}(a, b) = \Sigma\{\rho(x_i, y_i) : i \leq n\}$.*

8. *$\hat{\rho} = \bar{\rho} = \rho^*$.*

The following assertion is obvious.

Proposition 2.4.1. *Let ρ be a pseudo-quasi-metric on X and $\mathcal{V}$ be a non-Burnside quasivariety of topological monoids. For any $a{=}a_1a_2...a_n \in F^a(X, \mathcal{V})$ we put $a^{\leftarrow} = a_n...a_2a_1$. Then $a^{\leftarrow} \in F^a(X, \mathcal{V})$, $\rho^*(a, b){=}\rho(a^{\leftarrow}, b^{\leftarrow})$ and $(ab)^{\leftarrow}{=}b^{\leftarrow}a^{\leftarrow}$ for all $a, b \in F^a(X, \mathcal{V})$.*

Remark 2.4.1. *Invariant pseudo-metrics on free groups were constructed by M. I. Graev [98]. Stable metrics on free algebras were considered in [52]. Invariant quasi-metrics on free groups were constructed in [67] and [163].*

Remark 2.4.2. *Let A be a non-empty set and $\mathcal{V}$ be the non-Burnside quasivariety of all topological monoids. Consider that $\varepsilon \notin A$ and $X = A \cup \{\varepsilon\}$. Let $\rho(x, x) = 0$ and $\rho(x, y) = 1$ for all distinct points $x, y \in X$. Then $L(A) = F(X, \mathcal{V})$ is the family of all strings on the alphabet A. In this case there exists the maximal invariant extension $\hat{\rho}$ of ρ on $L(A)$. The metric $\hat{\rho}$ was studied in [55, 56, 57]. It was proved that the metric $\hat{\rho}$ coincides with the V. I. Levenshtein metric on $L(A)$ [130].*

2.5. Strongly invariant quasi-metrics

Fix non-Burnside quasivariety of topological monoids $\mathcal{V}$ and a space X with basepoint p_X.

Consider on X some linear ordering for which $p_X \leq x$ for any $x \in X$. On X consider the following distances ρ_l, ρ_r, ρ_s, where $\rho_l(x,x) = \rho_r(x,x) = 0$ for any $x \in X$; if $x, y \in X$ and $x < y$, then $\rho_l(x,y) = 1$, $\rho_l(y,x) = 0$, $\rho_r(x,y) = 0$, $\rho_r(y,x) = 1$, $\rho_s(x,y) = \rho_l(x,y) + \rho_r(x,y)$. By construction, ρ_l and ρ_r are quasi-metrics and ρ_s is a metric on X. Then $\rho_l^*(x,y)$ and $\rho_r^*(x,y)$ are invariant discrete quasi-metrics on $F(X,\mathcal{V})$ and ρ_s^* is a discrete invariant metric on $F(X,\mathcal{V})$. We consider this metric below.

A distance d on a semigroup G is strongly invariant if $d(xz, yz) = d(zx, zy) = d(x,y)$ for all $x, y, z \in G$.

On a group any invariant pseudo-quasi-metric is strongly invariant. For monoids that fact is not true.

Example 2.5.1. *Consider a semigroup $H = \{e, a, b\}$, where $ex = xe = x$ for each $x \in H$ and $xy = a$ provided $e \notin \{x, y\} \subset H$. The discrete metric d on H such that $d(x,y) = 0$ for $x = y$ and $d(x,y) = 1$ for $x \neq y$ is invariant on H and is not strongly invariant, since $0 = d(a,a) = d(ab,bb) = d(ba,bb) < d(a,b) = 1$. Let $\mathcal{W}(H)$ be the complete variety of topological monoids generated by the monoid H. For every monoid $G \in \mathcal{W}(H)$ there exists a unique point $a_G \in G$ such that $xy = a_G$ provided that $e \notin \{x,y\}$. Let X be a space with the basepoint p_X, $|X| \geq 2$ and ρ be a metric on X such that $\rho(x,y) = 1$ for all distinct points $x, y \in X$. Then ρ^* is an invariant metric on $F(X,\mathcal{W}(H))$ and $\rho^*(x,y) \geq 1$ for all distinct points $x, y \in F(X,\mathcal{W}(H))$. Let $c \in X \subseteq F(X,\mathcal{W}(H))$ and $c \neq p_X = e$. Then $c^2 \in F(X,\mathcal{W}(H))$ and $c^2 \neq c$. We have that $c^n = c^3 = c^2$ for any $n \geq 3$. Hence $1 \leq \rho^*(c,c^2)$ and $0 = \rho^*(c^2,c^2) = \rho^*(c^2,c^3) = \rho^*(c \cdot c, c^2 \cdot c) < \rho^*(c,c^2)$. In $F(X,\mathcal{W}(H))$ there exists a point $a \neq e$ such that $xy = a$ provided $e \notin \{x,y\}$. Hence the metric ρ^* is not strongly invariant on $F(X,\mathcal{W}(H))$. We observe that $\mathcal{W}(H)$ is a Burnside variety of the exponent (3,2). The above considerations permit to state that on the free monoid $F(X,\mathcal{W}(H))$ any invariant quasi-metric is not strongly invariant.*

For any pseudo-distance d S. Nedev [147] considered the adjoint pseudo-distance d^a defined by $d^a(x,y) = d(y,x)$.

Two properties $\mathcal{P}_1$ and $\mathcal{P}_2$ are called adjoint properties if the pseudo-distance d on a space X has property $\mathcal{P}_1$ if and only if the adjoint pseudo-distance d^a on a space X has property $\mathcal{P}_2$. If $\mathcal{P}_1 = \mathcal{P}_2$ and the properties $\mathcal{P}_1$ and $\mathcal{P}_2$ are adjoint, then we say that the property $\mathcal{P}_1$ is auto-adjoint.

Remark 2.5.1. *The auto-adjoint properties are the conditions for pseudo-distance to be invariant or strongly invariant on a semigroup G.*

The proof of the following assertion is simple.

Proposition 2.5.1. *Let $\mathcal{V}$ be a non-trivial quasivariety of topological monoids, ρ be a pseudo-distance on a space X with basepoint p_X. If $d = \rho^a$, then $d^* = \rho^{*a}$, i.e. $\rho^{a*} = \rho^{*a}$.*

The quasivariety of topological monoids $\mathcal{V}$ is rigid if for any space X, any word $a \in F(X, \mathcal{V})$, any point $c \in X \setminus \{p_x\}$ and any representation $ac = x_1x_2...x_n$, where $x_1, x_2, ..., x_n \in X$, there exists $m \leq n$ such that $x_m = c$ and $a = x_1x_2...x_{m-1}$. In this case $x_i = p_X = e$ for each $i > m$.

The variety of all topological monoids is rigid.

Theorem 2.5.1. *Let $\mathcal{V}$ be a non-Burnside rigid quasivariety of topological monoids, ρ be a quasi-metric on a space X with basepoint p_X and $\rho(x, p_X) = \rho(y, p_X)$ for all $x, y \in X \setminus \{p_X\}$, or $\rho(p_X, x) = \rho(p_X, y)$ for all $x, y \in X \setminus \{p_X\}$. Then $\rho^*(ac, bc) = \rho^*(ca, cb) = \rho^*(a, b)$ for all $a, b, c \in F(X, \mathcal{V})$.*

Proof. Assume that $\rho(p_X, x) = \rho(p_X, y)$ for all $x, y \in X \setminus \{p_X\}$. It is sufficient to prove the assertion of the theorem for $c \in X$. Assume that $\rho^*(ac, bc) = r < \rho^*(a, b)$, where $a, b \in F(X, \mathcal{V})$ and $c \in A$. Then, by definition, there exist the representations $ac = x_1x_2 \cdots x_n$ and $bc = y_1y_2 \cdots y_n$ such that $\rho^*(ac, bc) = \Sigma\{d(x_i, y_i) : i \leq p\}$.

From the definition of rigidity, there exist $p, q \leq n$ such that $x_p = y_q = c$, $a = x_1x_2...x_{p-1}$, $b = y_1y_2...y_{q-1}$ and $x_i = y_j = p_X$ with $p < i \leq n$ and $q < j \leq n$. We can assume that $n = max\{p, q\}$.

Case 1. $n = p = q$.

In this case $a = x_1x_2 \cdots x_{n-1}$, $b = y_1y_2 \cdots y_{n-1}$ and $\rho^*(a, b) \leq \Sigma\{d(x_i, y_i) : i \leq n-1\} = \Sigma\{d(x_i, y_i) : i \leq n\} = \rho^*(ac, bc) < \rho^*(a, b)$, a contradiction.

Case 2. $q < p = n$.

Then $y_n = p_X$, $x_n = y_q = c$, $a = x_1x_2 \cdots x_{n-1}$, $b = y_1y_2...y_{q-1} = y'_1y'_2...y'_{n-1}$, where $y'_j = y_j$ for $j < q$ and $y'_j = p_X$ for $j \geq q$. Since $\rho(x_q, p_X) \leq \rho(x_q, c) + \rho(c, p_X)$, we have $\rho^*(a, b) \leq \Sigma\{d(x_i, y'_i) : i \leq n-1\} \leq \Sigma\{d(x_i, y_i) : i \leq n\} = \rho^*(ac, bc) < \rho^*(a, b)$, a contradiction.

Case 3: $p < q = n$.

Then $x_n = p_X$, $y_n = x_p = c$, $a = x_1x_2 \cdots x_{p-1} = x'_1x'_2...x'_{n-1}$, $b = y_1y_2...y_{n-1}$, where $x'_i = x_i$ for $i < p$ and $x'_i = p_X$ for $i \geq p$. Since $\rho(p_X, y_p) \leq \rho(p_X, c)$, we have $\rho^*(a, b) \leq \Sigma\{d(x'_i, y_i) : i \leq n-1\} \leq \Sigma\{d(x_i, y_i) : i \leq n\} = \rho^*(ac, bc) < \rho^*(a, b)$, a contradiction.

Therefore, we proved that $\rho^*(ac, bc) = \rho^*(a, b)$ for all $a, b, c \in F(X, \mathcal{V})$. By virtue of Proposition 2.4.1, we have $\rho^*(ca, cb) = \rho^*(a^{\leftarrow}c^{\leftarrow}, b^{\leftarrow}c^{\leftarrow}) = \rho^*(a^{\leftarrow}, b^{\leftarrow}) = \rho^*(a, b)$ for all $a, b, c \in F(X, \mathcal{V})$.

Since the properties "$\rho(x, p_X) = \rho(y, p_X)$ for all $x, y \in X \setminus \{p_X\}$" and "$\rho(p_X, x) = \rho(p_X, y)$ for all $x, y \in X \setminus \{p_X\}$" are adjoint, the proof is complete. □

Corollary 2.5.1. *Let $\mathcal{V}$ be the non-Burnside rigid quasivariety of topological monoids, the space X is linear ordered such that $p_X \leq x$ for any $x \in X$. If $\rho \in \{\rho_l, \rho_r, \rho_s\}$, then ρ^* is a strongly invariant quasi-metric on $F(X, \mathcal{V})$.*

The following question is open.

Problem 2.5.1. *Does Theorem 2.5.1 hold for any non-Burnside quasivariety of topological monoids?*

2.6. Free monoids of T_0-spaces

Suppose that X is a topological space. Let x and y be points in X. We say that x and y can be separated by a function if there exists a continuous function $f : X \to [0, 1]$ into the unit interval such that $f(x) = 0$ and $f(y) = 1$.

A functionally Hausdorff space is a space in which any two distinct points can be separated by a continuous function.

The pseudo-distance d is continuous on a space X if any d-open subset $U \in \mathcal{T}(d)$ is open in X.

Lemma 2.6.1. *Let Y be a non-empty finite subspace of a T_0-space X. Then on X there exists a continuous pseudo-quasi-metric d_Y such that d_Y on Y generates the topology of the subspace Y.*

Proof. There exists a finite minimal family $\{U_1, U_2, ..., U_n\}$ of open subsets of X such that $T = \{U_1 \cap Y, U_2 \cap Y, ..., U_n \cap Y\}$ is the topology of the subspace Y. For each $i \leq n$ we put $d_i(x, y) = 1$ for $x \in U_i$, $y \in X \setminus U_i$ and $d_i(x, y) = 0$ for $x \in X \setminus U_i$ or $y \in U_i$. Then d_i is a continuous pseudo-quasi-metric on X and $\mathcal{T}(d_i) = \{\emptyset, U_i, X\}$. Hence $d_Y(x, y) = max\{d_i(x, y) : i \leq n\}$ is the desired pseudo-quasi-metric on X. □

The following theorem improves Theorem 2.3.3 and solves Problem 2.1.2 for complete non-Burnside quasivarieties of topological monoids.

Theorem 2.6.1. *Let $\mathcal{V}$ be a non-trivial complete non-Burnside quasivariety of topological monoids. Then:*

1. For each T_0-space X on the free monoid $F^a(X, \mathcal{V})$ there exists a T_0-topology $\mathcal{T}(qm)$ such that:

– $(F^a(X, \mathcal{V}), \mathcal{T}(qm)) \in \mathcal{V}$;

– X is a subspace of the space $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$;

– the topology $\mathcal{T}(qm)$ is generated by the family of all invariant pseudo-quasi-metrics on $F^a(X, \mathcal{V})$ which are continuous on X.

2. For each T_0-space X the free topological monoid $F(X, \mathcal{V})$ exists and is abstract free.

3. A space X is a T_1-space if and only if spaces $F(X, \mathcal{V})$ and $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ are T_1-spaces.

4. A space X is functionally Hausdorff if and only if the spaces $F(X, \mathcal{V})$ and $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ are functionally Hausdorff.

Proof. Fix a T_0-space X. Let $Q(X)$ be the family of all continuous pseudo-quasi-metrics on X and $IQ(X)$ be the family of all invariant pseudo-quasi-metrics on $(F^a(X, \mathcal{V}))$ which are continuous on X. Then $\mathcal{T}(qm)$ is the topology on $(F^a(X, \mathcal{V}))$ generated by the pseudo-quasi-metrics $IQ(X)$.

Claim 1. *X is a subspace of the space $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$.*

By virtue of Theorem 2.4.1, for each $\rho \in Q(X)$ we have $\hat{\rho} \in IQ(X)$ and $\rho(x, y) = \hat{\rho}(x, y)$ for all $x, y \in X$. Hence the pseudometrics $Q(X)$ and $IQ(X)$ generate on X the same topology. By virtue of Lemma 2.6.1, the topology of the space X is generated by the family of all continuous pseudo-quasi-metrics $Q(X)$. Hence X is a subspace of the space $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$.

Claim 2. *$(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ is a T_0-space.*

Fix two distinct points $a, b \in F^a(X, \mathcal{V})$. Let Y be a finite subspace of X such that $p_X \in Y$ and $a, b \in F^a(Y, \mathcal{V}) \subseteq F^a(X, \mathcal{V})$. By virtue of Lemma 2.6.1, on X there exists a continuous pseudo-quasi-metric d_Y which is a quasi-metric on Y. From the assertion 4 of Theorem 2.4.1 it follows that $\hat{d}_Y$ is a quasi-metric on $F^a(Y, \mathcal{V})$. Hence $\hat{d}_Y(a, b) + \hat{d}_Y(b, a) > 0$. Therefore $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ is a T_0-space.

Claim 3. *The topology $\mathcal{T}(qm)$ is generated by the family of all invariant pseudo-quasi-metrics $F^a(X, \mathcal{V})$ which are continuous on X.*

That assertion follows from the definition of the topology $\mathcal{T}(qm)$.

Claim 4. $(F^a(X, \mathcal{V}), \mathcal{T}(qm)) \in \mathcal{V}$.

Since the topology $\mathcal{T}(qm)$ is generated by the invariant pseudo-quasi-metrics, $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ is a a topological monoid. Hence the assertion of Claim 4 follows from Claim 2 and completeness of the quasivariety $\mathcal{V}$.

Claim 5. *For the T_0-space X the free topological monoid $F(X, \mathcal{V})$ is abstract free.*

Let G be the topological monoid $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$. There exists a continuous homomorphism $h : F(X, \mathcal{V}) \longrightarrow G$ such that $h(x) = x$ for each $x \in X$. Since G is abstract free relatively to X, h is a continuous isomorphism. Claim 5 is proved.

Claim 6. *A space X is a T_1-space if and only if the spaces $F(X, \mathcal{V})$ and $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ are T_1-spaces.*

If $F(X, \mathcal{V})$ is a T_1-space, then X is a T_1-space as a subspace of T_1-space. If $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ is a T_1-space, then $F(X, \mathcal{V})$ is a T_1-space, since $F(X, \mathcal{V})$ admits a continuous isomorphism onto $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$.

Assume now that X is a T_1-space. Fix two distinct points $a, b \in F^a(X, \mathcal{V})$. Let Y be a finite subspace of X such that $p_X \in Y$ and $a, b \in F^a(Y, \mathcal{V}) \subseteq F^a(X, \mathcal{V})$. By virtue of Lemma 2.6.1, on X there exists a continuous pseudo-quasi-metric d_Y which is a discrete metric on Y. Then $\hat{d}_Y$ is a discrete metric on $F^a(Y, \mathcal{V})$ and $F^a(Y, \mathcal{V})$ is a discrete subspace of $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$. Hence $\{a, b\}$ is a discrete subspace and $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ is a T_1-space. Claim 6 is proved.

Claim 7. *Let Y be a finite subspace of the functionally Hausdorff space X and $p_X \in Y$.*

Then there exists $d \in IQ(X)$ such that d is a pseudo-metric and $d(a, b) \geq 1$ for all distinct points $a, b \in F^a(Y, \mathcal{V})$.

Let $\{(x_i, y_i) : i \leq n\}$ be the family of all ordered pairs $x, y \in Y$ such that $x \neq y$. For any $i \leq n$ fix a continuous function $f_i : X \longrightarrow [0, 1]$ such that $h_i(x_i) = 0$ and $h_i(y_i) = 1$. Then $r_Y(x, y) = min\{1, \Sigma\{|f_i(x) - f_i(y)| : i \leq n\}\}$ is a continuous pseudo-metric on X and $r_Y(x, y) = 1$ for any two distinct points $x, y \in Y$. Then $\hat{r_Y}$ is the desired pseudo-metric from $IQ(X)$.

Claim 8. *The space X is functionally Hausdorff if and only if the spaces $F(X, \mathcal{V})$ and $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ are functionally Hausdorff.*

If $F(X, \mathcal{V})$ is a functionally Hausdorff space, then X is a T_1-space as a subspace of a functionally Hausdorff space. If $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ is a functionally Hausdorff space, then $F(X, \mathcal{V})$ is a functionally Hausdorff space, since $F(X, \mathcal{V})$ admits a continuous isomorphism onto $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$.

Assume now that X is a functionally Hausdorff space. Fix two distinct points $a, b \in F^a(X, \mathcal{V})$. Assume that $Y = Sup(a, b) = \{x_1, x_2, ..., x_n\}$, where $x_i \neq x_j$ for $i \neq j$. Since X is functionally Hausdorff space, there exists a construction function $f : X \longrightarrow [0, 1]$ such that $f(x_i) \neq f(x_j)$ for $i \neq j$. Consider the continuous pseudo-metric $\rho(x, y) = |f(x) - f(y)|$, $x, y \in X$. We have $\rho(x_i, y_i) \neq 0$ for $i \neq j$. Hence ρ is a metric on Y. Then ρ^* is a continuous pseudo-metric on $F^a(X, \mathcal{V})$, and ρ^* is a metric on $F^a(Y, \mathcal{V})$. Hence $\rho^*(a, b) \neq 0$. The function $g(x) = \rho^*(a, x)$ is continuous on $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$, $g(a) = 0$ and $g(b) \neq 0$. The function f is continuous on the space $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$, $f(a) = 0$ and $f(b) = 1$. Hence $(F^a(X, \mathcal{V}), \mathcal{T}(qm))$ is a functionally Hausdorff space. The Claim 8 and Theorem 2.6.1 are proved. □

Corollary 2.6.1. *Let $\mathcal{V}$ be a complete non-trivial quasivariety of topological monoids. Then for each completely regular space X:*

– on the free monoid $F^a(X, \mathcal{V})$ there exists a completely regular topology $\mathcal{T}(m)$ generated by a family of invariant pseudo-metrics such that $(F^a(X, \mathcal{V}), \mathcal{T}(m)) \in \mathcal{V}$, X is a subspace of the space $(F^a(X, \mathcal{V}), \mathcal{T}(m))$;

– the free topological monoid $F(X, \mathcal{V})$ exists, it is a functionally Hausdorff space and abstract free.

The following question is open.

Problem 2.6.1. *Let $\mathcal{V}$ be a non-trivial quasivariety of topological monoids. Under which conditions for a space X the free topological monoid $F(X, \mathcal{V})$ is a Hausdorff space, or a regular space, or a completely regular space?*

Remark 2.6.1. *Let X be a T_0-space and $\mathcal{V}$ be a non-trivial complete non-Burnside quasivariety of topological monoids. Then on $F(X, \mathcal{V})$ there exist:*

– the free topology $\mathcal{T}(f)$ such that $(F(x, \mathcal{V}), \mathcal{T}(f))$ is the free monoid of the space X in the quasivariety $\mathcal{V}$;

– the topology $\mathcal{T}(qm)$ generated by the invariant continuous pseudo-quasi-metrics on $(F(x, \mathcal{V}), \mathcal{T}(f))$;

– the topology $\mathcal{T}(m)$ generated by the invariant continuous pseudo-metrics on $(F(x, \mathcal{V}), \mathcal{T}(f))$.

These topologies satisfy the following properties:

P1. $\mathcal{T}(m) \subset \mathcal{T}(qm) \subset \mathcal{T}(f)$.

P2. $(F(x, \mathcal{V}), \mathcal{T}(m)), (F(x, \mathcal{V}), \mathcal{T}(f)) \in \mathcal{V}$.

P3. $(F(x, \mathcal{V}), \mathcal{T}(m)) \in \mathcal{V}$ *if and only if X is a functionally Hausdorff space.*

If the point p_X is isolated in X and $\mathcal{V}$ is the variety of all topological monoids, then on $F(X, \mathcal{V})$ we have $\mathcal{T}(qm) = \mathcal{T}(f)$. The invariant pseudo-metrics on topological groups were examined by G. Birkhoff [33] and Sh. Kakutani [120, 121]. There exists a locally compact topological group G with countable base without invariant metrics (see [120, 98, 99]). Since in G the involution $x \to x^{-1}$ is a homeomorphism, the topology of G is not generated by some family of invariant pseudo-quasi-metrics.

The following question is open.

Problem 2.6.2. *Let $\mathcal{V}$ be a non-trivial quasivariety of topological monoids. Under which conditions on $F(X, \mathcal{V})$ we have that $\mathcal{T}(qm) = \mathcal{T}(f)$?*

2.7. Free semi-topological monoids of T_0-spaces

A semi-topological semigroup is a semigroup with topology in which all translations $x \to ax$, $x \to xa$ are continuous.

A class $\mathcal{W}$ of semi-topological monoids is called a quasivariety of monoids if:

(F1) the class $\mathcal{W}$ is multiplicative;

(F2) if $G \in \mathcal{W}$ and A is a submonoid of G, then $A \in \mathcal{V}$;

(F3) every space $G \in \mathcal{W}$ is a T_0-space.

A class $\mathcal{W}$ of semi-topological monoids is called a complete quasivariety of monoids if it is a quasivariety with the next property:

(F4) if $G \in \mathcal{V}$ and T is a T_0-topology on G such that (G, T) is a semi-topological monoid, then $(G, T) \in \mathcal{V}$ too.

A quasivariety $\mathcal{V}$ of topological monoids is non-trivial if $|G| \geq 2$ for some $G \in \mathcal{V}$.

Let X be a non-empty topological space with a basepoint p_X and $\mathcal{W}$ be a quasivariety of topological monoids.

A free monoid of a space X in a class $\mathcal{W}$ is a semi-topological monoid $F(X, \mathcal{W})$ with the properties:

– $X \subseteq F(X, \mathcal{V}) \in \mathcal{W}$ and p_X is the unity of $F(X, \mathcal{V})$;

– the set X generates the monoid $F(X, \mathcal{V})$;

– for any continuous mapping $f : X \longrightarrow G \in \mathcal{V}$, where $f(p_X) = e$, there exists a unique continuous homomorphism $\bar{f} : F(X, \mathcal{V}) \longrightarrow G$ such that $f = \bar{f}|X$.

The abstract free monoid $F^a(X, \mathcal{W})$ of a space X in a class $\mathcal{W}$ is defined for quasivarieties of topological monoids.

Theorem 2.7.1. *Let $\mathcal{W}$ be a non-trivial quasivariety of semi-topological monoids. Then for each space X the following assertions are equivalent:*

1. There exists $G \in \mathcal{W}$ such that X is a subspace of G and p_X is the neutral element in G.

2. For the space X there exists the unique free semi-topological monoid $F(X, \mathcal{W})$.

Proof. Is similar to the proof of Theorem 2.1.1. □

Corollary 2.7.1. *Let $\mathcal{W}$ be a non-trivial quasivariety of semi-topological monoids. Then for each space X there exists the unique abstract free monoid $F^a(X, \mathcal{W})$.*

Let $\mathcal{W}$ be a non-trivial quasivariety of semi-topological monoids.

We put $\mathcal{W}_t = \{G \in \mathcal{W} : G \textit{ is a topological monoid}\}$. Obviously, $\mathcal{W}_t$ is a quasivariety of topological monoids.

Fix a space X for which there exists the free semi-topological monoid $F(X, \mathcal{W})$. Then there exists a unique continuous homomorphism $\lambda_X : F^a(X, \mathcal{V}) \longrightarrow F(X, \mathcal{V})$ such that $\lambda_X(x) = x$ for each $x \in X$. The monoid $F(X, \mathcal{W})$ is called abstract free if λ_X is a continuous isomorphism.

Theorem 2.7.2. *Let $\mathcal{W}$ be a non-trivial non-Burnside quasivariety of semi-topological monoids. Then for each space X the following assertions are equivalent:*

1. The class $\mathcal{W}_t$ is a non-trivial non-Burnside quasivariety of topological monoids.

2. For each space X we have $F^a(X, \mathcal{W}) = F^a(X, \mathcal{W}_t)$.

3. For each T_0-space X on the free monoid $F^a(X, \mathcal{W})$ there exists a T_0-topology $\mathcal{T}(qm)$ such that:

– $(F^a(X, \mathcal{V}), \mathcal{T}(qm)) \in \mathcal{W}_t \subseteq \mathcal{W}$;

– X is a subspace of the space $(F^a(X, \mathcal{W}), \mathcal{T}(qm))$;

– the topology $\mathcal{T}(qm)$ is generated by the family of all invariant pseudo-quasi-metrics on $F^a(X, \mathcal{V})$ which are continuous on X.

4. For each T_0-space X there exists the free topological monoid $F(X, \mathcal{W})$ and it is abstract free. Also, there exists a continuous isomorphism $\mu_X : F(X, \mathcal{W}) \longrightarrow F(X, \mathcal{W}_t)$ such that $\mu_X(x) = x$ for each $x \in X$.

5. A space X is a T_1-space if and only if spaces $F(X, \mathcal{W})$ and $(F^a(X, \mathcal{W}), \mathcal{T}(qm))$ are T_1-spaces.

6. A space X is functionally Hausdorff if and only if the spaces $F(X, \mathcal{W})$ and $(F^a(X, \mathcal{W}), \mathcal{T}(qm))$ are functionally Hausdorff.

Proof. Assertion 1 is obvious. For any space X denote by X_t the set X with the discrete topology. Then $G_t \in \mathcal{W}_t$ for each $G \in \mathcal{W}$. Fix a T_0-space X. The space $F^a(X, \mathcal{W})$ is discrete. Hence $F^a(X, \mathcal{W}) \in \mathcal{W}_t$ and Assertion 2 is proved.

Assertion 3 follows from Assertion 2 and Theorem 2.6.1.

Assertions 4 - 6 follow from Assertion 3 and Theorem 2.6.1. □

Condition of completeness is essential.

Example 2.7.1. *Let B be the semigroup ω with the topology $T(B) = \{\emptyset, B\} \cup \{B \setminus F : F$ is a finite subset of $B\}$. Then B is a semi-topological monoid and B is not a topological monoid. Denote now by $W(B)$ the quasivariety generated by B. Then the elements of $W(B)$ are the submonoids of the monoids of the form B^M. Thus any non-trivial monoid $G \in W(B)$ is not a topological monoid. Therefore the class $W(B)_t$ is trivial.*

2.8. On topological digital spaces

A space X is called an Alexandroff space if it is a T_0-space and the intersection of any family of open sets is open [6].

Alexandroff spaces were first introduced in 1937 by P. S. Alexandroff [6] (see also [12]) under the name discrete spaces, where he provided the characterizations in terms of sets and neighbourhoods.

If (X, T) is an Alexandroff space, then we say that T is a T_0-discrete topology.

We observe the importance of distances with natural values. We affirm that this fact is important from topological point of view as well.

Theorem 2.8.1. *On a space X there exists a quasi-metric with the natural values if and only if X is an Alexandroff space.*

Proof. Let X be an Alexandroff space. For any $x \in X$ denote by M_x the intersection of all open sets which contains x. Then M_x is the minimal open set which contains the point $x \in X$. Observe that if $x, y \in X$, $x \neq y$, and $y \in M_x$, then $M_y \subset M_x$ and $x \notin M_y$. Consider the distance $\rho(x, y)$, where $\rho(x, x) = 0$ for any $x \in X$, $\rho(x, y) = 0$ if $y \in M_x$, and $\rho(x, y) = 1$ if $y \notin M_x$. We affirm that ρ is a quasi-metric with natural values. By construction, $\rho(x, y) \in \{0, 1\}$ and ρ has natural values. Let $x, y, z \in X$. If $\rho(x, y) = \rho(y, z) = 0$, then $y \in M_x$ and $z \in M_y \subset M_x$. Hence $\rho(x, z) = 0$. In this case $\rho(x, y) + \rho(y, z) = \rho(x, z)$. If $\rho(x, y) + \rho(y, z) \geq 1$, then $\rho(x, z) \leq 1$ and $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$. Therefore ρ is a quasi-metric.

If d is a quasi-metric on X with natural values, then $M_x = \{y \in X : d(x, y) < 1\}$ is the minimal open set which contains the point $x \in X$. Therefore $(X, T(d))$ is an Alexandroff space, and this concludes the proof of Theorem 2.8.1. □

General criteria of quasi-metrizability of spaces were proved in [147].

Let $\leq$ be a partial ordering on a set X. For any point $x \in X$ we put $M(x, \leq) = \{y \in X : x \leq y\}$. Then $\{M(x, \leq) : x \in X\}$ is a base of the T_0-discrete topology $T(\leq)$ on X.

Let T be a T_0-topology on a set X. For any points $x, y \in X$ we put $x \leq_T y$ if and only if $x \in cl_X\{y\}$. Then $\leq_T$ is a partial ordering on X. By construction, $\leq = \leq_{T(\leq)}$, $T \subset T(\leq_T)$ and $T = T(\leq_T)$ if and only if T is T_0-discrete topology (see [6]).

For any T_0-topology T on X we put $aT = T(\leq_T)$. If $M(x) = \cap\{U \in T : x \in U\}$, then $\{M(x) : x \in X\}$ is the minimal base of the topology aT. We say that aT is the Alexandroff modification of the topology T.

The following assertion is obvious.

Proposition 2.8.1. *Let T be a T_0-topology on a set X. Then aT is the unique T_0-discrete topology on the space X such that $\leq_T = \leq_{aT}$. Moreover, $\leq_T = \leq_{T'}$ for any intermediary topology $T \subset T' \subset aT$.*

Theorem 2.8.2. *Let (G, T) be a topological semigroup. Then (G, aT) is a topological semigroup too.*

Proof. We put $M(x) = \cap\{U \in T : x \in U\}$. Then $\{M(x) : x \in X\}$ is the base of the topology aT and $M(x) \cdot M(y) \subset M(x \cdot y)$. The proof is complete. □

Corollary 2.8.1. *Let $\mathcal{V}$ be a non-trivial complete non-Burnside quasivariety of topological monoids. Then for each space X the following assertions are equivalent:*

1. *$F(X, \mathcal{V})$ is an Alexandroff space.*
2. *On a space $F(X, \mathcal{V})$ there exists a quasi-metric with the natural values.*
3. *X is an Alexandroff space.*

Proposition 2.8.2. *Let G be a topological semigroup and X be a connected subspace of G. If X algebraically generates the semigroup G, then G is a connected space.*

Proof. For each $n \in \mathbb{N}$ we put $G_n(X) = \{x_1 \cdot x_2 \cdot \ldots \cdot x_n : x_1, x_2, \ldots x_n \in X\}$. By construction, the subspace $G_n(X)$ of G is connected as a continuous image of the connected space X^n and $G_n(X) \subset G_{n+1}(X)$. Hence $G = \cup\{G_n(X) : n \in \mathbb{N}\}$ is a connected space. The proof is complete. □

A digital space is a pair (D, α), where D is a non-empty set and α is a binary, symmetric relation on D such that for any two elements $x, y \in D$ there is a finite sequence $\{x_0, x_1, \ldots, x_n\}$ of elements in D such that $x = x_0$, $y = x_n$ and $(x_j, x_{j+1}) \in \alpha$ for $j \in 0, 1, \ldots, n-1\}$.

The topological methods may be applied in the study of reflexive or anti-reflexive binary structures. We develop that point of view for reflexive digital structures.

Let ρ be a distance on the non-empty set D. We consider that $(x, y) \in \alpha_\rho$ if and only if $\rho(x, y) \cdot \rho(y, x) = 0$. We say that α_ρ is the binary relation generated by the distance ρ.

A binary relation α on the set D is compatible with the topology T on D if T is a T_0-topology and $(x, y) \in \alpha$ if and only if $x \in cl_{(X,T)}\{y\}$ or $x \in cl_{(X,T)}\{y\}$.

Proposition 2.8.3. *If a binary relation α on the set D is compatible with the topology T on D, then the binary relation α is compatible by the T_0-discrete topology aT.*

Proof. For any $x \in D$ denote $M_x = \cap\{U \in T : x \in U\}$. Let T_a be the topology on D generated by the open base $\{M_x : x \in D\}$. Then M_x is the minimal open set from T_a which contains the point $x \in X$. It is obvious that $x \in cl_{(X,T)}\{y\}$ if and only if $x \in cl_{(X,aT)}\{y\}$. The proof is complete. □

Proposition 2.8.4. *Let a symmetric binary relation α on the non-empty set D is compatible with the T_0-discrete topology T on D. The following assertions are equivalent:*

1. *(D, α) is a digital space.*
2. *(D, T) is a connected space.*
3. *There exists a discrete quasi-metric ρ on D such that $\alpha = \alpha_\rho$ and the space $(D, T(\rho))$ is connected.*

Proof. Implication 1 → 2 follows from Proposition 2.8.3. Implication 2 → 3 → 2 follows from Theorem 2.8.1.

Assume that (D, T) is a connected Alexandroff space.

For any $x \in D$ denote by $M_1(x)$ the intersection of all open sets which contains x. Let $M_{n+1}(x) = \cup\{M_1(y) : M_1(y) \cap M_n(x) \neq \emptyset\}$ and $M_\omega(x) = \cup\{M_n(x) : n \in \mathbb{N}\}$.

By construction, if $y \in M_1(x)$, then $(x, y) \in \alpha$. Hence, if $y \in M_n(x)$, then there is a sequence $\{x_0, x_1, ..., x_n\}$ of elements in D such that $x = x_0$, $y = x_n$ and $(x_j, x_{j+1}) \in \alpha$ for $j \in \{0, 1, ..., n-1\}$.

Fix $x \in D$. We affirm that the set $M_\omega(x)$ is closed. If the set $M_\omega(x)$ is not closed, then there exists a point $y \in cl_X M_\omega(x) \setminus M_\omega(x)$. Hence $M_1(y) \cap M_\omega(x) \neq \emptyset$. In this case $M_1(y) \cap M_n(x) \neq \emptyset$ for some $n \in \mathbb{N}$ and $y \in M_{n+1}(x) \neq \emptyset$, a contradiction. Thus the set $M_\omega(x)$ is non-empty and open-and-closed. Since (X, T) is a connected space, we have $M_\omega(x) = X$. Therefore (D, α) is a digital space. Implication 2 → 1 is proved. The proof is complete. □

If the digital structure α on a set D is compatible with a T_0-discrete topology T on D, then we say that (D, T) is a topological digital space and put $(D, \alpha) \equiv (D, T)$. Otherwise the digital space (D, α) is not topological. Hence a topological space X is a topological digital space if and only if X is a connected Alexandroff space (see [114, 122, 123]).

From Corollary 2.8.1 and Propositions 2.8.2 and 2.8.4 follows:

Corollary 2.8.2. *Let $\mathcal{V}$ be a non-trivial complete non-Burnside quasivariety of topological monoids. Then for each space X the following assertions are equivalent:*

1. *$F(X, \mathcal{V})$ is a topological digital space.*
2. *X is a topological digital space.*

There exists non-topologically digital spaces (D, α) (see [114]). For example, let $D = \{a, b, c, d, e\}$ and $\alpha = \{(a, a), (a, b), (b, a), (b, b), (b, c), (c, b), (c, c), (c, d), (d, c),$
$(d, d), (d, e), (e, d), (e, e), (e, a), (a, e)\}$. Then the digital space (D, α) is not topological.

If D is a non-empty set and $\alpha = D \times D$, then (D, α) is a digital space such that for any linear ordering $\leq$ on D we have $\alpha = b(\leq)$ and binary relation α is compatible with the topology $T((\leq)$. We observe that a topology is compatible with a unique binary structure and a binary structure may be compatible with a set of arbitrary cardinality of topologies.

Now let α be an anti-reflexive digital structure on G. Let ρ be a distance on the non-empty set D. We consider that $(x, y) \in \alpha_\rho$ if and only if $x \neq y$ and $\rho(x, y) \cdot \rho(y, x) = 0$. We say that α_ρ is the binary relation generated by the distance ρ. A binary anti-reflexive relation α on the set D is compatible with the topology T on D if T is a T_0-topology and $(x, y) \in \alpha$ if and only if $x \neq y$ and $x \in cl_{(X,T)}\{y\}$ or $x \in cl_{(X,T)}\{y\}$. For anti-reflexive digital structures similar assertions hold as in the reflexive case.

2.9. Conclusions for chapter 2

Free objects in a given class of algebras of same type (groups, rings, semigroups, monoids, etc.) play an important role in algebra and theoretical computer science. According to definition, a language is a subset of a free monoid over a given alphabet. Some information can be represented as an element of the given free monoid. In this chapter:

- were defined non-Burnside quasi varieties of topological monoids;
- for each quasivariety for any space free topological and free abstract monoids are defined.

If $\mathcal{V}$ is the class of all topological monoids, then:

- $\mathcal{V}$ is a rigid quasivariety of topological monoids;
- $F(X, \mathcal{V}) = F^a(X, \mathcal{V})$ is the space of all canonical strings on the alphabet X.

Free monoids research permits to obtain the following generel conclusions:

1. For each non-trivial quasivariety $\mathcal{V}$ of topological monoids and any non-empty space X was proposed a method of construction of the abstract free monoid $F^a(X, \mathcal{V})$.
2. Was established that for any non-Burnside quasivariety $\mathcal{V}$ and any quasi-metric ρ on a set X with basepoint p_X on free monoid $F^a(X, \mathcal{V})$ there exists a unique stable quasi-metric $\hat{\rho}$ with the properties:
   (a) $\rho(x, y) = \hat{\rho}(x, y)$ for all $x, y \in X$.

(b) If d is an invariant quasi-metric on $F^a(X, \mathcal{V})$ and $d(x, y) \leq \rho(x, y)$ for all $x, y \in X$, then $d(x, y) \leq \hat{\rho}(x, y)$ for all $x, y \in F^a(X, \mathcal{V})$.

(c) If ρ is a metric, then $\hat{\rho}$ is a metric as well.

(d) If $Y \subseteq X$, $d = \rho|Y$ and $\hat{d}$ is the maximal invariant extension of d on $F^a(Y, \mathcal{V})$, then $F^a(Y, \mathcal{V}) \subseteq F^a(X, \mathcal{V})$ and $\hat{d} = \hat{\rho}|F^a(Y, \mathcal{V})$.

(e) For any quasi-metric ρ on X and any points $a, b \in F^a(X, \mathcal{V})$ there exists $n \in N$ and representations $a = a_1a_2...a_n$, $b = b_1b_2...b_n$ such that $a_1, b_1, a_2, b_2, ..., a_n, b_n \in X$ and $\hat{\rho}(a, b) = \sum\{\rho(a_i, b_i) : i \leq n\}$.

3. Was introduced the notion of rigid non-Burnside quasivariety $\mathcal{V}$ and proved that for the extension $\hat{\rho}$ on $F^a(X, \mathcal{V})$ of the quasi-metric ρ on X we have $\hat{\rho}(a, b) = \hat{\rho}(ca, cb) = \hat{\rho}(ac, bc)$ for all $a, b, c \in F^a(X, \mathcal{V})$.

4. The method of extension of quasi-metrics on free monoids in the complete non-Burnside quasivariety of topological monoids permit:

   - to construct distinct admissible topologies of $F^a(X, \mathcal{V})$ for any T_0-space X;
   - to prove that the free topological monoid $F^a(X, \mathcal{V})$ exists for any space X;
   - to establish that the free topological monoid $F(X, \mathcal{V})$ is abstract free, i.e. is canonically isomorphic with the abstract free monoid $F^a(X, \mathcal{V})$.

   This fact solves problems posed by A. I. Maltsev in 1958 for free universal topological algebras. Similar results were obtained for quasivarieties of semi-topological monoids as well.

5. It was proved that if $\mathcal{V}$ is a complete quasivariety of topological monoids, then :

   - X is an Alexandroff space if and only if $F(X, \mathcal{V})$ is an Alexandroff space;
   - X is a digital space if and only if $F(X, \mathcal{V})$ is a digital space.

3. MEASURES OF SIMILARITY ON MONOIDS OF STRINGS

In this chapter we continue with the study of the results obtained in the previous chapter, mainly the extension method of a quasi-metric on a free monoid, and apply them on the monoid of strings on an alphabet A. This allows to introduce the notion of parallel decomposition, which leads to new interesting results related to measure of similarity, efficiency and penalty of alignment of a pair of strings, as well as relations between the former. Furthermore, properties of Hamming, Levenshtein and Graev-Markov distances are analyzed.

The examined properties and the results obtained in this chapter are published in the articles [36, 37, 38, 43, 55, 56, 57, 58, 59, 62, 63] and serve as a foundation for the next chapter. The mentioned results can also be applied in various problems related to similarity between sequences of characters.

3.1. Monoid of strings on alphabet A

A monoid is a semigroup with an identity element. Fix a non-empty set A. The set A is called an alphabet. We put $\bar{A} = A \cup \{\varepsilon\}$. Let $L^*(A)$ be the set of all finite strings $a_1a_2 \dots a_n$ with $a_1, a_2, \dots, a_n \in \bar{A}$. Let ε be the empty string. Consider the strings $a_1a_2 \dots a_n$ for which $a_i = \varepsilon$ for some $i \leq n$. If $a_i \neq \varepsilon$, for any $i \leq n$ or $n = 1$ and $a_1 = \varepsilon$, the string $a_1a_2 \dots a_n$ is called an *irreducible string* or *canonical string*. The set $Supp(a_1a_2 \dots a_n) = \{a_1, a_2, \dots, a_n\} \cap A$ is the support of the string $a_1a_2 \dots a_n$ and $l(a_1a_2 \dots a_n) = |\{i \leq n : a_i \neq \varepsilon\}|$ is the length of the string $a_1a_2 \dots a_n$. For two strings $a_1 \dots a_n$ and $b_1 \dots b_m$, their product (concatenation) is $a_1 \dots a_nb_1 \dots b_m$. If $n \geq 2$, $i < n$ and $a_i = \varepsilon$, then the strings $a_1 \dots a_n$ and $a_1 \dots a_{i-1}a_{i+1} \dots a_n$ are considered equivalent. In this case any string is equivalent to one unique canonical string. Two strings a and b are called equivalent, denoted $a \sim b$, if a and b are equivalent to the same canonical string. We identify the equivalent strings and $\kappa : L^*(A) \longrightarrow L(A)$ is the operation of the identification. The set $L(A)$ of all canonical strings is the family of all classes of equivalent strings. In this case $L^*(A)$ is a semigroup and $L(A)$ becomes a monoid with identity ε. The set $L(A)$ is not a subsemigroup of $L^*(A)$. Only the set $L(A) \setminus \{\varepsilon\}$ is a subsemigroup of the semigroup $L^*(A)$.

Let $Supp(a, b) = Supp(a) \cup Supp(b) \cup \{\varepsilon\}$, and $Supp(a, a) = Supp(a) \cup \{\varepsilon\}$. It is well known that any subset $L \subset L(A)$ is an abstract language over the alphabet A.

Remark 3.1.1. *If $\mathcal{V}$ is the variety of all monoids, then $L(A) = F^a(A, \mathcal{V})$. Thus, we can apply the results of Chapter 2 to study the monoids of strings $L(A)$ on the alphabet A.*

Let a, b be two strings. For any two representations $a = a_1a_2 \dots a_n$ and $b = b_1b_2 \dots b_m$ we put

$$d_H(a_1a_2\dots a_n, b_1b_2\dots b_m) = |\{i : a_i \neq b_i, i \leq min\{n,m\}\}| \\ +|\{i : n < i \leq m, b_i \neq e_i\}| \\ +|\{j : m < j \leq n, a_j \neq e_i\}|.$$

The function d_H is called the Hamming distance on the space of strings [106, 55, 56].

Now we put:

$$d_G(a,b) = inf\{d_H(a,b) : a = a_1a_2\dots a_n, b = b_1b_2\dots b_n\}.$$

The function d_G is called the Graev – Markov distance on the space of strings [98, 135].

Remark 3.1.2. *Let $\rho(x,x) = 0$ and $\rho(x,y) = 1$ for all $x, y \in \bar{A}$ and $x \neq y$. Then ρ is the discrete metric on $\bar{A}$. Therefore $d_G = \rho^*$ on $L(A)$.*

The V. I. Levenshtein's distance $d_L(a,b)$ between two strings $a = a_1a_2\dots a_n$ and $b = b_1b_2\dots b_m$ is defined as the minimum number of insertions, deletions, and substitutions required to transform one string into the other [130, 56].

We put $A^{-1} = \{a^{-1} : a \in A\}$, $\varepsilon^{-1} = \varepsilon$, $(a^{-1})^{-1} = a$ for any $a \in A$ and consider that $A^{-1} \cap \bar{A} = \emptyset$. Denote $\check{A} = A \cup A^{-1} \cup \{\varepsilon\}$. Let $\check{L}(A) = L(\check{A})$ be the set of all strings over the set $\check{A}$. The strings over the set $\check{A}$ are called *words*. A word $a = a_1a_2\dots a_n \in \check{L}(A)$ is called an irreducible string if either $n = 1$ and $a_1 \in \check{A}$, or $n \geq 2$, $a_i \neq \varepsilon$ for any $i \leq n$ and $a_j^{-1} \neq a_{j+1}$ for each $j < n$.

Let $a = a_1a_2\dots a_n \in \check{L}(A)$ and $n \geq 2$. Then:

- if $i \leq n$ and $a_i = \varepsilon$, then the words $a_1\dots a_n$ and $a_1\dots a_{i-1}a_{i+1}\dots a_n$ are considered equivalent;

- if $i < n$ and $a_i^{-1} = a_{i+1}$, then the words $a_1a_2\dots a_n$ and $a_1\dots a_{i-1}\varepsilon a_{i+2}\dots a_n$ are considered equivalent.

In this case, any word $a_1a_2\dots a_n \in \check{L}(A)$ is equivalent to one unique irreducible word from $\check{L}(A)$. We identify equivalent words. Classes of equivalence form free group $F(A)$ over A with unity ε. We have that $L(A)$ is a subsemigroup of the group $F(A)$.

Let $a = a_1a_2\dots a_n \in F(A)$ be an irreducible word. The representation $a = x_1x_2\dots x_m \in L^*(A)$ is called an *almost irreducible* representation of a if there exist $1 \leq i_1 < i_2 < \dots < i_n \leq m$ such that $a_j = x_{i_j}$ for any $j \leq n$ and $x_i = \varepsilon$ for each $i \in \{1,2,\dots,m\} \setminus \{i_1, i_2, \dots, i_n\}$. If $a = a_1a_2\dots a_n \in L^*(A)$ is a representation of the string a, then $a_1a_2\dots a_n$ is an almost irreducible word.

If $a = a_1a_2\dots a_n$, then $a^s = a_na_{n-1}\dots a_1$ and $a^{-1} = a_n^{-1}a_{n-1}^{-1}\dots a_1^{-1}$. The word a^s is the symmetric word of a and a^{-1} is the inverse word of a. If a and b are equivalent words, then the words a^{-1} and b^{-1} are equivalent, as well as the words a^s and b^s.

Hence the mappings $\cdot^s, \cdot^{-1} : F(A) \longrightarrow F(A)$ are the group automorphisms. Obviously that $L(A)^s = L(A)$.

Let $a, b \in A$ and $a \neq b$, then we put $d_H(a, b) = d_H(a^{-1}, b^{-1}) = d_H(a, \varepsilon) = d_H(\varepsilon, a) = d_H(a^{-1}, \varepsilon)$ $= d_H(\varepsilon, a^{-1}) = 1$. If $a \in A$ and $b \in A^{-1}$, then $d_H(a, b) = d_H(b, a) = 2$. For any $x \in \check{A}$ we put $d_H(x, x)$ $= 0$. Thus d_H is a metric on $\check{A}$. For any two words $a_1a_2 \ldots a_n, b_1b_2 \ldots b_m \in \check{L}(A)$ we put:

$$\begin{aligned} d_H(a_1a_2 \ldots a_n, b_1b_2 \ldots b_m) =& \Sigma\{d_H(a_i, b_i) : i \leq min\{n, m\}\} \\ &+|\{i : n < i \leq m, b_i \neq e_i\}| \\ &+|\{j : m < j \leq n, a_j \neq e_i\}|. \end{aligned}$$

For $a, b \in F(A)$ we put:

$$\check{d}(a, b) = inf\{d_H(a, b) : a = a_1 \ldots a_n \in \check{L}(A), b = b_1 \ldots b_n \in \check{L}(A)\}.$$

Remark 3.1.1. *The function $\check{d}$ is called the Graev – Markov distance on the free group [98]. The method of extensions of distances for free groups, used by us, was proposed by A. A. Markov [135] and M. I. Graev [98]. For metrics on free universal algebras it was extended in [52], for quasi-metrics on free groups and varieties of groups it was examined in [163, 67].*

M. I. Graev [98] has proved the following assertions:

G1. $\check{d}$ is an invariant metric on $F(A)$ and $\check{d}(a, b) = d_H(a, b)$ for all $a, b \in A^*$.

G2. If ρ is an invariant metric on $F(A)$ and $\rho(x, y) \leq d_H(x, y)$ for any $x, y\ inA^*$, then $\rho(x, y) \leq \check{d}(x, y)$ for any $x, y \in F(A)$.

G3. For any two words $a, b \in F(A)$ there exist $m \geq 1$ and two almost irreducible representations $a = x_1x_2 \ldots x_m$ and $b = y_1y_2 \ldots y_m$ such that $\check{d}(a, b) = d_H(x_1x_2 \ldots x_m, y_1y_2 \ldots y_m)$.

Theorem 3.1.1. *The distance d_G on a monoid $L(A)$ has the following properties:*

1. d_G is a strong invariant metric on $L(A)$ and $d_G(x, y) = d_G(zx, zy) = d_G(xz, yz)$ for all $x, y, z \in L(A)$.

2. $d_G(a, b) = d_G(a^s, b^s)$ for all $a, b \in L(A)$.

3. If ρ is an invariant metric on $L(A)$ and $\rho(x, y) \leq d_G(x, y)$ for all $x, y \in \bar{A}$, then $\rho(a, b) \leq d_G(a, b)$ for all $a, b \in L(A)$.

4. For any $a, b \in L(A)$ there exist $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_n \in Supp(a, a)$ and $y_1, y_2, \ldots, y_n \in Supp(b, b)$ such that $a = x_1x_2 \ldots x_n$, $b = y_1y_2 \ldots y_n$ such that $n \leq l(a) + l(b)$ and $d_G(a, b) = |\{i : i \leq n, a_i \neq b_i\}| = d_H(x_1x_2 \ldots x_n, y_1y_2 \ldots y_n)$.

5. $d_G(a, b) = \check{d}(a, b) \leq d_H(a, b)$ for all $a, b \in L(A)$.

Proof. Fix $a, b \in L(A)$. Let $a = a_1a_2 \ldots a_n$, $b = b_1b_2 \ldots b_n$. If $n > l(a) + l(b)$, then there exists $i \leq n$ such that $a_i = b_i = \varepsilon$, $a = a_1a_2 \ldots a_{i-1}a_{i+1} \ldots a_n$, $b = b_1b_2 \ldots b_{i-1}b_{i+1}, \ldots b_n$ and

$d_H(a_1a_2...a_n, b_1b_2...b_n) = d_H(a_1...a_{i-1}a_{i+1}...a_n, b_1...b_{i-1}b_{i+1}...b_n)$. Hence $d_G(a,b) = inf\{d_H(a_1a_2\dots a_n,\ b_1b_2\dots b_n) : a = a_1a_2\dots a_n\ ,\ b = b_1b_2\dots b_n,\ n \le l(a)+l(b)\}$. Since we have finite pairs of parallel representations $a = a_1a_2\dots a_m$, $b = b_1b_2\dots b_m$ of length $m \le l(a)+l(b)$, there exist $n \in \mathbb{N}$, $x_1, x_2, \dots, x_n \in Supp(a,a)$ and $y_1, y_2, \dots, y_n \in Supp(b,b)$ such that $a = x_1x_2\dots x_n$, $b = y_1y_2\dots y_n$ with $n \le l(a)+l(b)$ and $d_G(a,b) = |\{i : i \le n, a_i \ne b_i\}| = d_H(x_1x_2\dots x_n, y_1y_2\dots y_n)$. Thus, *Assertion 4* is proved. *Assertion 2* is obvious.

Fix $a, b \in L(A)$ and $c \in A$. It is clear that $d_G(ca,cb) \le d_G(a,b)$. Assume that $d_G(ca,cb) < d_G(a,b)$. Then there exist representations $ca = x_1x_2\dots x_n$ and $cb = y_1y_2\dots y_n$ such that $n \le l(a)+l(b)+2$ and $d_G(ca,cb) = d_H(x_1x_2\dots x_n, y_1y_2\dots y_n)$, where $A \cap \{x_i, y_i\} \ne \emptyset$ for each $i \le n$. If $x_1 = y_1$, then $x_1 = y_1 = c$. In this case $a = x_2\dots x_n$, $b = y_2\dots y_n$ and $d_G(a,b) \le d_H(x_2\dots x_n, y_2\dots y_n) = d_H(x_1x_2\dots x_n, y_1y_2\dots y_n) = d_H(ca,cb) < d_H(a,b)$, a contradiction. Hence $x_1 \ne y_1$. In this case we have two possibilities: $x_1 = c$, $y_1 = \varepsilon$ or $x_1 = \varepsilon$, $y_1 = c$. We can assume that $x_1 = c$ and $y_1 = \varepsilon$. Let $1 < j$, $y_j = c$ and $y_i = \varepsilon$ for each $i < j$. We put $u_1 = v_i = \varepsilon$ for each $i \le j$, $u_i = x_i$ for each $i \ge 2$ and $v_k = y_k$ for each $k > j$. Then $b = u_1u_2\dots u_n$, $b = v_1v_2\dots v_n$, $0 = d_H(u_1,v_1) < d_H(x_1,y_1) = 1$, $d_H(x_j,y_j) \le 1$, $d_H(u_j,v_j) \le 1$ and $d_H(u_i,v_i) = d_H(x_i,y_i)$ for $i \in \{2,3,...,j-1,j+1,...,n\}$. Hence $d_G(a,b) \le d_H(u_1u_2...u_n, v_1v_2...v_n) \le d_H(x_1x_2...x_n, y_1y_2...y_n)$ $= d_G(ca,cb) < d(a,b)$, a contradiction. Hence $d_G(ca,cb) = d(a,b)$. From *Assertion 2* it follows that $d_G(ac,bc) = d_G(a,b)$. *Assertion 1* is proved.

We put $d(x,x) = 0$ and $d(x,y) = 1$ for any distinct strings $x, y \in L(A)$. Let $ID(A)$ denote the family of all invariant metrics ρ on $L(A)$ with the property: $\rho(x,y) \le d(x,y)$ for all $x, y \in \bar{(}A)$. Since $d \in ID(A)$, the set $ID(A)$ is non-empty. Now we put $d^*(a,b) = sup\{\rho(a,b) : \rho \in ID(A)\}$. One can easily observe that $d^* \in ID(A)$, $d(a,b) \le d^*(a,b)$ for any $a, b \in L(A)$ and $d(x,y) = d^*(x,y)$ $= 1$ for all distinct $x, y \in \bar{(}A)$.

Property 1. If $\rho \in ID(A)$, then

$$\begin{aligned}\rho(x_1x_2\dots x_n, y_1y_2\dots y_n) \le& |\{i \le 1 : x_i \ne y_i\}|\\ =& d_H(x_1x_2\dots x_n, y_1y_2\dots y_n)\end{aligned}$$

for any two strings $(x_1x_2\dots x_n, y_1y_2\dots y_n) \in L(A)$.

This property follows from the conditions of invariance of metric d.

Property 2. $d_G = d^* = d_L$.

Since d_G and d^* are invariant distances on $L(A)$ and they are constructed with the conditions of extremity

$$d^*(a,b) = sup\{\rho(a,b) : \rho \in ID(A)\},$$
$$d_G(a,b) = inf\{d_H(a,b) : a = a_1a_2\dots a_n\ ,\ b = b_1b_2\dots b_n\},$$

we have $d_G = d^*$. In the following section of this chapter it was proved that $d^* = d_L$. The equality

$d_G(a,b) = \check{d}(a,b)$ for all $a, b \in L(A)$ follows from the Graev's assertion $G3$ in the above Remark. This completes the proof of the theorem. □

Example 3.1.1. *The metrics d, $d_G = d_L = d^*$ are strong invariant on $L(A)$. On $L(A)$ there exists a metric $d_r \in ID(A)$ which is invariant, but not strong invariant. Fix a real number r for which $2^{-1} \leq r < 1$. We put $d_r(x,x) = 0$ for each $x \in L(A)$, $d(x,y) = r$ for any distinct strings $x, y \in L(A) \setminus \{\varepsilon\}$ and $d(0,x) = d(x,0) = 1$ for any $x \in L(A) \setminus \{\varepsilon\}$. Then d is an invariant metric on G. Fix $a \in A$. Since $r = d(a, aa) = d(\varepsilon \cdot a, a \cdot a) < d(\varepsilon, a) = 1$, the metric d_r is not strong invariant.*

Remark 3.1.2. *For the metric d_H we have $d_H(a,b) \leq max\{l(a, l(b)\}$ for any strings $a, b \in L(A)$. The Hamming distance d_H is left invariant: $d_H(xa, xb) = d(a,b)$ for all strings $x, a, b \in L(A)$. Assume now that $x, y, z \in A$, $a = xyzxyz$, b= $yzxy$ and $c = xyz$. Then $d_G(a,b) = 2$ and $6 = d_H(a,b) < d_H(ac, bc) = 9$. Therefore, metric d_H is not right invariant.*

3.2. Relations to Hamming and Levenshtein distances

If $a, b \in L(a,b)$ and $a = a_1 a_2 \cdots a_n$, $b = b_1 b_2 \cdots b_m$ are the canonical decompositions, then for $m \leq n$ the number

$$d_H(a,b) = d_H(b,a) = |\{i \leq m : a_i \neq b_i\}| + n - m$$

is called the *Hamming distance* [106] between strings a and b.

The *Levenshtein distance* [130] between two strings $a = a_1 a_2 \cdots a_n$ and $b = b_1 b_2 \cdots b_m$ is defined as the minimum number of insertions, deletions, and substitutions required to transform one string to the other. A formal definition of Levenshtein's distance $d_L(a,b)$ is given by the following formula:

$$d_L(a_1 \cdots a_i, b_1 \cdots b_j) = \begin{cases} i, & \text{if j=0,} \\ j, & \text{if i=0,} \\ \min \begin{cases} d_L(a_1 \cdots a_{i-1}, b_1 \cdots b_j) + 1 \\ d_L(a_1 \cdots a_i, b_1 \cdots b_{j-1}) + 1 \\ d_L(a_1 \cdots a_{i-1}, b_1 \cdots b_{j-1}) + 1_{(a_i \neq b_j)}, \end{cases} \end{cases}$$

where $1_{(a_i \neq b_j)}$ equals to 0 if $a_i = b_j$ and to 1 otherwise.

Theorem 3.2.1. *$d_L(a,b) = \rho^*(a,b) \leq d_H(a,b)$ for any $a, b \in L(A)$.*

Proof. To prove the equality $d_L(a,b) = \rho^*(a,b)$, we will first prove that $d_L(a,b) \leq \rho^*(a,b)$, and then that $d_L(a,b) \geq \rho^*(a,b)$.

We begin with the observation that the parallel decompositions of two strings a, b allow more transparent evaluation of the Levenshtein distance $d_L(a, b)$. If $a = v_1u_1v_2u_2 \cdots v_n$ and $b = w_1u_1w_2u_2 \cdots w_n$ are optimal parallel decompostions, then for transformation of b to a it is sufficient to transform any w_i to v_i. The cost of transformation of w_i to v_i is $\leq \max\{l(w_i), l(v_i)\}$. Hence $d_L(a, b) \leq \rho^*(a, b)$.

The proof of the inequality $d_L(a, b) \geq \rho^*(a, b)$ is based on the Levenshtein distance formula, as well as the construction of the transformation of string a to string b. We observe that the Levenshtein distance is calculated recursively using the *memoization* matrix and *dynamic programming* technique [77, p. 359–378]. A small snapshot of the memoization matrix calculation is presented below.

Table 1: Construction of memoization matrix for Levenshtein distance

Diag	Above
Left	min(Above + delete, Left + insert, Diag + $1_{a_i \neq b_j}$)

Distance d_L calculated on subtrings $a_1 \cdots a_i$ of string a and substring $b_1 \cdots b_j$ of string b is equal to the minimum of the following values:

- $d_L(a_1 \cdots a_{i-1}, b_1 \cdots b_j) + 1,$ (1)
- $d_L(a_1 \cdots a_i, b_1 \cdots b_{j-1}) + 1,$ (2)
- $d_L(a_1 \cdots a_{i-1}, b_1 \cdots b_{j-1}) + 1_{a_i \neq b_j}.$ (3)

Remark : the operation (1) is the delete operation, (2) is the insert operation, and (3) is the substitution operation.

Once all of the above values are calculated and the memoization matrix is filled, the distance is given by the value in the cell on the n^{th} row and m^{th} column.

The construction of the transformation of string a into string b is based on the values of the memoization matrix. At each point of the construction process, we will execute operations on both strings a and b, and obtain another pair of strings a' and b' equivalent to the initial pair a and b. We use the top-down analysis approach to describe the transformation process step by step. The process below starts with $i = n, j = m, p = 0, q = 0$ and both a', b' as empty strings:

- if when calculating $d_L(a_1 \cdots a_i, b_1 \cdots b_j)$ we used operation (1), then we deleted a character from string a at position i, which is equivalent to inserting the ε character in string b at the corresponding position. In this case, in the building process of a' and b', we put

$p := p+1$, $v'_p = \{a_i\}$,$w'_p = \{\varepsilon\}$, $a' := v'_p \cup a'$, $b' := w'_p \cup b'$. Next, we proceed to calculate $d_L(a_1 \cdots a_{i-1}, b_1 \cdots b_j)$.

- if when calculating $d_L(a_1 \cdots a_i, b_1 \cdots b_j)$ we used operation (2), then we inserted the ε character in string a at position i. In this case, in the building process of a' and b', we put $p := p+1$, $v'_p = \{\varepsilon\}$,$w'_p = \{b_j\}$, $a' := v'_p \cup a'$, $b' := w'_p \cup b'$. Next, we proceed to calculate $d_L(a_1 \cdots a_i, b_1 \cdots b_{j-1})$.

- if when calculating $d_L(a_1 \cdots a_i, b_1 \cdots b_j)$ we used operation (3), then we either substituted the character at position i of string a with the character at position j of string b, or we did not make any change in case if $a_i = b_j$. If $a_i = b_j$, we put $q =: q+1$,$u'_q = \{a_i\}$, $a' := u'_q \cup a'$, $b' := u'_q \cup b'$. If $a_i \neq b_j$, we put $p =: p+1$, $v'_p = \{a_i\}$, $w'_p = \{b_j\}$, $a' := v'_p \cup a'$, $b' := w'_p \cup b'$. Next, we proceed to calculate $d_L(a_1 \cdots a_{i-1}, b_1 \cdots b_{j-1})$.

According to the above steps, we observe that string a' is equivalent to string a, and string b' is equivalent to b by construction. But, we also have that the decomposition $a' = v'_p u'_q v'_{p-1} u'_{q-1} \cdots u'_1 v'_1$ and $a' = w'_p u'_q w'_{p-1} u'_{q-1} \cdots u'_1 w'_1$ obtained from the above construction process, represent a parallel decomposition of strings a and b. Thus, we have that $d_L(a,b) = E(a,b) \geq \rho^*(a,b)$. This completes the proof of the equality $d_L(a,b) = \rho^*(a,b)$.

We will now prove the second part of the theorem, namely that $\rho^*(a,b) \leq d_H(a,b)$. Let $d_H(a,b) < max\{l(a), l(b)\} = n$, where $n = l(a) \geq l(b) = m$. Then $a = a_1 a_2 \cdots a_n$, $b = b_1 b_2 \cdots b_m$, $a_i \neq \varepsilon$ for any $i \leq n$, and or $m = 1$ and $b_1 = \varepsilon$, or $b_j \neq \varepsilon$ for any $j \leq m$. In this case $d_H(a,b) = n - |\{i \leq m : a_i = b_i\}|$ and we have the representations $a = (a_1)(a_2)\cdots(a_m)(a_{m+1}\cdots a_n)$ and $b = (b_1)(b_2)\cdots(b_m)(\varepsilon)$ which generate two parallel decompositions α, β with $E(\alpha,\beta) = d_H(a,b)$. Therefore $\rho^*(a,b) \leq E(\alpha,\beta) = d_H(a,b)$. The proof is complete.

□

Corollary 3.2.1. *Distance d_L is strictly invariant, i.e. $d_L(ac,bc) = d_L(ca,cb) = d_L(a,b)$ for any $a,b,c \in L(A)$.*

Remark 3.2.1. *The Hamming distance d_H is not invariant.*

Example 3.2.1. *Let $n = m+p$ and strings $a = (01)^n$, $b = (10)^m$, $c = (01)^p$. We obtain the following distance values for the above strings:*

$$d_L(a,b) = 2p, \rho^*(a,b) = 2p, d_H(a,b) = 2n,$$
$$d_L(ac,bc) = 2p, \rho^*(ac,bc) = 2p, d_H(ac,bc) = 2n.$$

Remark 3.2.2. *If $l(a) = l(b)$, then $d_H(ac,bc) = d_H(a,b)$ for any $a,b,c \in L(A)$. Additionally, the following equality always holds:*

$$d_H(ca, cb) = d_H(a, b).$$

3.3. Efficiency and penalty of two strings

The longest common substring and pattern matching in two or more strings is a well known class of problems. For any two strings $a, b \in L(A)$ we find the decompositions of the form $a = v_1u_1v_2u_2 \dots v_ku_kv_{k+1}$ and $b = w_1u_1w_2u_2 \dots w_ku_kw_{k+1}$, which can be represented as $a = a_1a_2 \dots a_n$, $b = b_1b_2 \dots b_n$ with the following properties:

- some a_i and b_j may be empty strings, i.e. $a_i = \varepsilon$, $b_j = \varepsilon$;

- if $a_i = \varepsilon$, then $b_i \neq \varepsilon$, and if $b_j = \varepsilon$, then $a_j \neq \varepsilon$;

- if $u_1 = \varepsilon$, then $a = v_1$ and $b = w_1$;

- if $u_1 \neq \varepsilon$, then there exists a sequence $1 \leq i_1 \leq j_1 < i_2 \leq j_2 < \dots < i_k \leq j_k \leq n$ such that: $u_1 = a_{i_1} \dots a_{j_1} = b_{i_1} \dots b_{j_1}$, $u_2 = a_{i_2} \dots a_{j_2} = b_{i_2} \dots b_{j_2}$, $u_k = a_{i_k} \dots a_{j_k} = b_{i_k} \dots b_{j_k}$;

- if $v_1 = w_1 = \varepsilon$, then $i_1 = 1$;

- if $v_{k+1} = w_{k+1} = \varepsilon$, then $j_k = n$;

- if $k \geq 2$, then for any $i \in \{2, \dots, k\}$ we have $v_i \neq \varepsilon$ or $w_i \neq \varepsilon$.

In this case $l(u_1) + l(u_2) + \dots + l(u_k) = |\{i : a_i = b_i\}|$.

The above decompositions forms are called *parallel decompositions* of strings a and b [55, 56, 57]. For any parallel decompositions $a = v_1u_1 \dots v_ku_kv_{k+1}$ and $b = w_1u_1 \dots w_ku_kw_{k+1}$ the number

$$\begin{aligned} &E(v_1u_1 \dots v_ku_kv_{k+1}, w_1u_1 \dots w_ku_kw_{k+1}) \\ &\quad = \sum_{i \leq k+1} \{\max\{l(v_i), l(w_i)\}\} = d_H(x_1x_2 \dots x_n, y_1y_2 \dots y_n) \end{aligned}$$

is called the efficiency of the given parallel decompositions. The number $E(a, b)$ is equal to the minimum of efficiency values of all parallel decompositions of the strings a, b and is called the *common efficiency of the strings a,b*. It is obvious that $E(a, b)$ is well determined and $E(a, b) = d_G(a, b)$. We say that the parallel decompositions $a = v_1u_1v_2u_2 \dots v_ku_kv_{k+1}$ and $b = w_1u_1w_2u_2 \dots w_ku_kw_{k+1}$ are optimal if the following equality holds:

$$E(v_1u_1v_2u_2 \dots v_ku_kv_{k+1}, w_1u_1w_2u_2 \dots w_ku_kw_{k+1}) = E(a, b).$$

This type of decompositions are associated with the problem of approximate string matching [146]. If the decompositions $a = v_1u_1 \dots v_ku_kv_{k+1}$ and $b = w_1u_1 \dots w_ku_kw_{k+1}$ are optimal and $k \geq 2$, then we may consider that $u_i \neq \varepsilon$ for any $i \leq k$.

Any parallel decompositions $a = a_1a_2 \dots a_n = v_1u_1v_2u_2 \dots v_ku_kv_{k+1}$ and $b = b_1b_2 \dots b_n =$

$w_1u_1w_2u_2\ldots w_ku_kw_{k+1}$ generate a common sub-sequence $u_1u_2\ldots u_k$. The number

$$m(a_1a_2\ldots a_n, b_1b_2\ldots b_n) = l(u_1) + l(u_2) + \ldots + l(u_k)$$

is the *measure of similarity* of the decompositions [27, 151]. There exist parallel decompositions $a = v_1u_1v_2u_2\ldots v_ku_kv_{k+1}$ and $b = w_1u_1w_2u_2\ldots w_ku_kw_{k+1}$ for which the measure of similarity is maximal. The maximum value of the measure of similarity of all decompositions is denoted by $m^*(a,b)$. The maximum value of the measure of similarity of all optimal decompositions is denoted by $m^\omega(a,b)$. We can note that $m^\omega(a,b) \le m^*(a,b)$. For any two parallel decompositions $a = a_1a_2\ldots a_n$ and $b = b_1b_2\ldots b_n$ as in [56], we define the *penalty factors* as

$$p_r(a_1a_2\ldots a_n, b_1b_2\ldots b_n) = |\{i \le n : a_i = \varepsilon\}|,$$

$$p_l(a_1a_2\ldots a_n, b_1b_2\ldots b_n) = |\{j \le n : b_j = \varepsilon\}|,$$

$$\begin{aligned} p(a_1a_2\ldots a_n, b_1b_2\ldots b_n) &= |\{i \le n : a_i = \varepsilon\}| + |\{j \le n : b_j = \varepsilon\}| \\ &= p_r(a_1a_2\ldots a_n, b_1b_2\ldots b_n) + p_l(a_1a_2\ldots a_n, b_1b_2\ldots b_n) \end{aligned}$$

and

$$\begin{aligned} &M_r(a_1a_2\ldots a_n, b_1b_2\ldots b_n) \\ &\quad = m(a_1a_2\ldots a_n, b_1b_2\ldots b_n) - p_r(a_1a_2\ldots a_n, b_1b_2\ldots b_n) \\ &M_l(a_1a_2\ldots a_n, b_1b_2\ldots b_n) \\ &\quad = m(a_1a_2\ldots a_n, b_1b_2\ldots b_n) - p_l(a_1a_2\ldots a_n, b_1b_2\ldots b_n) \\ &M(a_1a_2\ldots a_n, b_1b_2\ldots b_n) \\ &\quad = m(a_1a_2\ldots a_n, b_1b_2\ldots b_n) - p(a_1a_2\ldots a_n, b_1b_2\ldots b_n) \end{aligned}$$

as the *measures of proper similarity*.

The number $d_H(a_1a_2\ldots a_n, b_1b_2\ldots b_n) = |\{i \le n : a_i \ne b_i\}|$ is the Hamming distance between decompositions and it is another type of penalty. We have that

$$p(a_1\ldots a_n, b_1\ldots b_n) \le d_H(a_1\ldots a_n, b_1\ldots b_n).$$

The assertions from the following theorem establish the main results.

Theorem 3.3.1. *Let a and b be two non-empty strings, $a = a_1a_2\ldots a_n$ and $b = b_1b_2\ldots b_n$ be the initial optimal decompositions, and $a = a'_1a'_2\ldots a'_q$ and $b = b'_1b'_2\ldots b'_q$ be the second* *decompositions, which are arbitrary. Denote by*

$$
\begin{aligned}
m &= m(a_1a_2\dots a_n, b_1b_2\dots b_n), & m' &= m(a'_1a'_2\dots a'_q, b'_1b'_2\dots b'_q),\\
p &= p(a_1a_2\dots a_n, b_1b_2\dots b_n), & p' &= p(a'_1a'_2\dots a'_q, b'_1b'_2\dots b'_q),\\
p_l &= p_l(a_1a_2\dots a_n, b_1b_2\dots b_n), & p'_l &= p_l(a'_1a'_2\dots a'_q, b'_1b'_2\dots b'_q),\\
p_r &= p_r(a_1a_2\dots a_n, b_1b_2\dots b_n), & p'_r &= p_r(a'_1a'_2\dots a'_q, b'_1b'_2\dots b'_q),\\
r &= d_H(a_1a_2\dots a_n, b_1b_2\dots b_n), & r' &= d_H(a'_1a'_2\dots a'_q, b'_1b'_2\dots b'_q),
\end{aligned}
$$

$$
\begin{aligned}
M &= m - p, & M' &= m' - p',\\
M_l &= m - p_l, & M'_l &= m' - p'_l,\\
M_r &= m - p_r, & M'_r &= m' - p'_r.
\end{aligned}
$$

The following assertions are true:

1. $p' - p = 2(m' - m) + 2(r' - r)$.
2. *If the second decompositions are non optimal, then* $M_l > M'_l$ *and* $M_r > M'_r$.
3. *If the second decompositions are optimal, then* $M_l = M'_l$ *and* $M_r = M'_r$ *and the measures* M_l *and* M_r *are constant on the set of optimal parallel decompositions.*
4. *If* $m' \geq m$ *and the second decompositions are non optimal, then* $p' > p$, $p_{l'} > p_l$, $p'_r > p_r$ *and* $M > M'$.
5. *If* $m' = m$ *and the second decompositions are optimal, then* $p' = p$, $p_{l'} = p_l$, $p'_r = p_r$ *and* $M' = M$.
6. *If* $m' \leq m$ *and the second decompositions are non optimal, then* $m' - r' < m - r$.

The proof of Theorem 3.3.1 follows from the next lemmas.

Lemma 3.3.1.

$$
\begin{aligned}
p_r(a'_1a'_2\dots a'_q, b'_1b'_2\dots b'_q) &= q - l(a),\\
p_l(a'_1a'_2\dots a'_q, b'_1b'_2\dots b'_q) &= q - l(b),\\
p(a'_1a'_2\dots a'_q, b'_1b'_2\dots b'_q) &= 2q - l(a) - l(b).
\end{aligned}
$$

Proof. Follows immediately from the definitions of penalty factors and parallel decompositions. □

Lemma 3.3.2. $p' - p = 2(m' - m) + 2(r' - r)$.

Proof. From Lemma 3.3.1 it follows that $p' - p = (2q - l(a) - l(b)) - (2n - l(a) - l(b)) = 2(q - n)$. Since $q = m' + r'$ and $n = m + r$, the proof is complete. □

Lemma 3.3.3. $p'_l - p_l = p'_r - p_r = (m' - m) + (r' - r)$.

Proof. We can assume that $l(a) \leq l(b)$. Then $p_l = (l(b) - l(a)) + l_r$ and $p_l - p_r = l(b) - l(a)$. Hence $p_l - p_r = p'_l - p'_r$ and $p'_l - p_l = p'_r - p_r$. The equality $p' - p = (p'_r - p_r) + (p'_l - p_l)$ and Lemma 3.3.2 complete the proof. □

Lemma 3.3.4. *Assume that $m' > m$. Then:*

1. $M > M'$, $M_l \geq M'_l$ *and* $M_r \geq M'_r$.
2. $M_l > M'_l$ *and* $M_r > M'_r$ *provided that the second decompositions are non optimal.*
3. $M_l = M'_l$ *and* $M_r = M'_r$ *provided that the second decompositions are optimal.*

Proof. Since the initial decompositions are optimal, we have $r' \geq r$. Moreover, we have $r' = r$ if and only if the second decompositions are optimal as well. By virtue of definitions, we have $n = m + r$ and $q = m' + r'$. Therefore $n < q$. From Lemma 3.3.2, it follows that $p' - p = 2(m' - m) + 2(r' - r)$ and $p < p'$. Thus $p' - p > m' - m$ and $M = m - p > m' - p' = M'$.

Also, from Lemma 3.3.3, it follows that $p'_l - p_l = p'_r - p_r = (m' - m) + (r' - r)$. Hence, $M_l = m - p_l = (m' - p'_l) + (r' - r) = M'_l + (r' - r)$ and $M_r = m - p_r = (m' - p'_r) + (r' - r) = M'_r + (r' - r)$. Since $r' \geq r$ and $r' = r$ if and only if the second decompositions are optimal, the proof is complete. □

Corollary 3.3.1. *The measures M_l and M_r are constant on the set of optimal parallel decompositions.*

Lemma 3.3.5. *Let $m' = m$. Then:*

1. $M \geq M'$, $M_l \geq M'_l$ *and* $M_r \geq M'_r$.
2. $M_l > M'_l$ *and* $M_r > M'_r$ *provided that the second decompositions are non optimal.*
3. $M_l = M'_l$ *and* $M_r = M'_r$ *provided that the second decompositions are optimal.*

Proof. We have that $n = m + r$ and $q = m' + r'$. Since $r \leq r'$, we have that $n \leq q$.

Assume that $M < M'$. Then $m - p < m' - p'$, $p' = 2q - l(a) - l(b)$ and $p = 2n - l(a) - l(b)$. Hence $m - 2n + l(a) + l(b) < m - 2q + l(a) + l(b)$, or $-2n < -2q$ and $n > q$, a contradiction.

From Lemma 3.3.3 it follows that $p'_l - p_l = p'_r - p_r = r' - r$. Hence $p'_l \geq p_l$ and $p'_r \geq p_r$. If the second decompositions are non optimal, then $p'_l > p_l$ and $p'_r > p_r$. Assertions are proved. □

Lemma 3.3.6. *Assume that $m' < m$ and the second decompositions are non optimal. Then $M_l > M'_l$ and $M_r > M'_r$.*

Proof. Since the initial decompositions are optimal, we have $r' > r$. By virtue of Lemma 3.3.3, we have $p'_l - p_l = p'_r - p_r = (m' - m) + (r' - r)$. Hence, $M_l = m - p_l = (m' - p'_l) + (r' - r) = M'_l + (r' - r)$ and $M_r = m - p_r = (m' - p'_r) + (r' - r) = M'_r + (r' - r)$. Since $r' - r > 0$, the proof is complete. □

Remark 3.3.1. *From Assertions of Theorem 3.3.1 it follows that on the class of all optimal decompositions of given two strings:*

- *the maximal measure of proper similarity is attained on the optimal parallel decomposition with minimal penalties (minimal measure of similarity);*
- *the minimal measure of proper similarity is attained on the optimal parallel decomposition with maximal penalties (maximal measure of similarity).*

For any two non-empty strings there are parallel decompositions with maximal measure of similarity and optimal decompositions on which the measure of similarity is minimal.

We present below an example that illustrates the relations on proper similarities and penalties implied by Assertion 4 of Theorem 3.3.1.

Example 3.3.1. *Let*

$$A \quad A \quad A \quad A \quad C \quad C \quad C$$

$$C \quad C \quad C \quad B \quad B \quad B \quad B$$

be trivial optimal decompositions of strings a, b, *and*

$$\begin{matrix} A & A & A & A \\ \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{matrix} \left(\begin{matrix} C & C & C \\ \\ C & C & C \end{matrix} \right) \begin{matrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \\ B & B & B & B \end{matrix}$$

be their non-optimal decompositions. Then

$$m' = 3, r' = 8, p' = 8,$$

$$m = 0, r = 7, p = 0.$$

In this example we have that $-5 = m' - r' > m - r = -7$ *and* $-5 = m' - p' = M' < M = m - p = 0$.

The following example shows that there are some exotic non-optimal parallel decompositions $a = a'_1 a'_2 \cdots a'_q$ and $b = b'_1 b'_2 \cdots b'_q$, such that for optimal decompositions $a = a_1 a_2 \cdots a_n$ and $b = b_1 b_2 \cdots b_n$ we have $m' < m$, $p' < p$, and $M' > M$.

Example 3.3.2. *Let*

$$A \quad B \quad C \quad D \begin{pmatrix} E \\ \\ E \end{pmatrix} F$$
$$C \quad D \quad E \quad F \qquad\quad D$$

be trivial non-optimal decompositions of strings a, b and

$$\begin{matrix} A & B \\ \\ \varepsilon & \varepsilon \end{matrix} \begin{pmatrix} C & D & E & F \\ \\ C & D & E & F \end{pmatrix} \begin{matrix} \varepsilon & \varepsilon \\ \\ E & D \end{matrix}$$

be their optimal decompositions. Then

$$m' = 1, r' = 5, p' = 0,$$

$$m = 4, r = 4, p = 4.$$

We have that $m' - p' = M' > M = m - p$, and $m' - r' < m - r$.

The above examples show that Theorem 3.3.1 cannot be improved in the case of $m' < m$.

Decompositions with minimal penalty and maximal proper similarity are of significant interest. Moreover, if we solve the problem of text editing and correction, the optimal decompositions are more favorable. Therefore, the optimal decompositions are the best parallel decompositions and we may solve the string match problems only on class of optimal decompositions.

Remark 3.3.2. *The optimal decompositions:*

- *describe the proper similarity of two strings,*
- *permit to obtain long common sub-sequences,*
- *permit to calculate the distance between strings,*
- *permit to appreciate changeability of information over time.*

3.4. Computational algorithms of distances

Let A be a given alphabet and $\bar{A} = A \cup \varepsilon$. A (dis)similarity measure on a set $\bar{A}$ is a function of two variables $s : \bar{A} \times \bar{A} \longrightarrow R$, possibly subject to additional properties. A range similarity query centered at $a \in \bar{A}$ consists of all $x \in \bar{A}$ determined by the inequality $s(a, x) < k$ or $s(a, x) > k$, depending on the type of similarity measure. A similarity workload is a workload whose queries are generated by a similarity measure. The formula $d(a, b) = s(a, a) - s(a, b)$, $a, b \in A$ is a distance. In many cases $d(a, b)$ is a quasi-metric. By instance, applied to the similarity measure given by BLOSUM62, as well as to most other matrices from the BLOSUM family, d is a quasi-metric on A.

Hence any quasi-metric ρ on $\bar{A}$ may be considered as a similarity. The similarity ρ on $\bar{A}$ may be extended on the space of strings $L(A)$. If ρ is a quasi-metric on $\bar{A}$ then its extension on $L(A)$ is a quasi-metric ρ_1 on $L(A)$ such that $\rho_1(x, y) = \rho(x, y)$ for all $x, y \in \bar{A}$. Obviously, there exist many extensions of the given quasi-metric ρ on $\bar{A}$. In the Chapter 2 we proved that on $L(A)$ there exists the extension d_H of the Hamming distance and the extension d_L of Levenstein distance. For any extension ρ_1 on $L(A)$ of ρ on $\bar{A}$ are important the algorithms of calculating of the distance $\rho_1(a, b)$ for any two strings $a, b \in L(A)$. We say that the calculation of the distance $d_L(a, b)$ is determined by some parallel decompositions of the given two strings a and b. The decompositions $a = a_1 a_2 ... a_n$ and $b = b_1 b_2 ... b_m$ of the strings a, b are parallel if $n = m$. Let ρ be a quasi-metric on $\bar{A}$. The decompositions $a = a_1 a_2 ... a_n$ and $b = b_1 b_2 ... b_m$ of the strings a, b are called ρ-optimal if $n = m$ and $d_H(a, b) = \sum\{(\rho(a_i, b_i) : i \leq n\}$. Hence, are important the algorithms of calculation of the distance $d_H(a, b)$ and the algorithms of construction of all optimal pairs of parallel decompositions of the given two strings a and b.

The algorithm of computing the Levenshtein distance for the case of a discrete metric is well known. Below we show a well known algorithm that permits to calculate the Graev-Markov-Levenstein distance between two irreducible strings for any metric.

Algorithm 1: Metric:
Given $x, y \in F(A)$ compute $\check{d}(x, y)$, for the case of metric.

Data: $x = x_1 x_2 \ldots x_n$, $y = y_1 y_2 \ldots y_m$, metric function $\check{d}$ on $\check{A}$.
Parameters: costs of insertion and removal operations - $cost_{insert}$ and $cost_{remove}$ respectively.
Result: $d_L(x, y)$, and matrix D.
// initialize distance matrix
for $i \leftarrow 1$ **to** m **do** D[i,0]=i;
for $j \leftarrow 1$ **to** n **do** D[0,j]=j;
// initialize loop variables
$i := 1, j := 1$;
for $j \leftarrow 1$ **to** n **do**
  for $i \leftarrow 1$ **to** m **do**
    if $dist(x_i, y_j) = 0$ **then**
      d[i, j] := d[i-1, j-1];
    else
      // Dynamic Programming recursive function
      d[i,j] := min(d[i-1, j] + $cost_{remove}$, min(d[i, j-1] + $cost_{insert}$, d[i-1, j-1] + $dist(x_i, y_i)$));
return D[m, n], D;

For any two non-empty strings there exist the parallel decompositions with maximal measure of similarity and the optimal decompositions on which measure of similarity is minimal. The pseudo-code of such algorithm is presented below:

Algorithm 2: Maximal Measure of Similarity:
Finds maximum value of measure of similarity of $x, y \in L(\bar{A})$.

```
/* Helper functions to compute similarity and penalty factors */
Function similarity (a,b)
    n = max(length(a), length(b))
    sim = 0
    for i ← 1 to n do
        if (a[i] == b[i]) sim = sim + 1
    return sim;
Function penalty (a,b)
    n = max(length(a), length(b))
    pen = 0
    for i ← 1 to n do
        if (a[i] == ε) pen = pen + 1
        if (b[i] == ε) pen = pen + 1
    return pen;
/* Main Algorithm Body */
Data: x = x1x2 ... xn, y = y1y2 ... ym.
Result: Maximal measure of similarity of x and y.
// Calling Metric or Quasi-Metric Functions
d, D := metric(x, y);
// Calling BackTracking function BuildOPD
BuildOPD(n,m,x,y,a,b,D,S);
max_sim = 0
for ((a,b): S) do
    sim = similarity(a,b) - penalty(a,b)
    max_sim = max_sim < sim ? sim : max_sim
return max_sim
```

The above algorithm makes calls to function *BuildOPD*, which uses the memoization matrix to generates optimal parallel decompositions. This algorithm uses the memoization matrix $D[m, n]$ calculated in the previous algorithm. The idea is to traverse from the bottom right cell $D[m, n]$ to the top left cell $D[0, 0]$ and at each step to evaluate whether the minimal distance was obtained by replacement, deletion or insertion. The algorithm uses recursive *backtracking* to reconstruct all decompositions of strings a and b. The pseudocode for this function is presented below.

Algorithm 3: Optimal Parallel Decompositions (OPD):
Generate all optimal parallel decompositions of given $x, y \in L(A)$.

```
/* BackTracking Function Building Optimal Parallel Decompositions   */
Function BuildOPD (n,m; x,y,a,b; D, S)
    // n, m - current indexed in D matrix
    // x, y - input strings; a,b - building blocks of OPD
    // D - memoization matrix of x and y
    if (n = 0)and(m = 0) then
        S.append(reverse(a), reverse(b));
    else
        if ((n > 0)and(m > 0)) and ((D[n,m] = D[n-1,m-1] + dist(x_n, y_m)) or
          ((D[n,m] = D[n-1,m-1]) and (dist(x_n, y_m) = 0))) then
            BuildOPD(n-1,m-1,a+x_n,b+y_m, D, S);
        if (n > 0) and (D[n,m] = D[n-1,m] + cost_remove) then
            BuildOPD(n-1,m,a+x_n,b+ε, D, S);
        if (m > 0) and (D[n,m] = D[n,m-1] + cost_insert) then
            BuildOPD(n,m-1,a+ε,b+y_m, D, S);

/* Main Algorithm Body                                                */
Data: x = x_1 x_2 ... x_n, y = y_1 y_2 ... y_m.
Parameters: costs of insertion and removal operations - cost_insert and cost_remove
            respectively.
Result: Strings representing optimal parallel decompositions.
// Calling Levenshtein Distance or Quasi-Metric Functions
d, D := LevenshteinDistance(x, y);
// Initialize building blocks of OPD
a := ''; b := ''; S = [];
// Calling BackTracking function BuildOPD
BuildOPD(n,m,x,y,a,b,D, S);
```

In the worst case scenario its complexity is $O(m + n)$ (this happens when we separately traverse the matrix horizontally and vertically). This result is achieved with the help of prioritizing the direction of analysis when traversing the matrix. We first look to the north-west and only afterwards to the northern and western cell values. We stop the reconstruction process once the algorithm reaches the cell at $D[0, 0]$. The reasoning behind this decision is to find the most optimal decomposition among all possible decompositions of strings a and b.

3.5. General applications and examples

First and foremost let us look at how we can apply the above results in information distance problems such as string search, text correction, and pattern matching. We have presented one such example in the previous section – the edit distance.

We also mentioned the problem of DNA/RNA sequence alignment, which goes back as early as 1970 [151]. Applications in bioinformatics of the distance ρ^* include phylogenetic analysis, whole genome phylogeny, and detection of acceptable mutations [157]. Other applications using edit operations like insertions and editions in DNA can be found in [155].

We begin this section with the pseudo-codes of the decompositions alignment algorithm, which constructs the shortest optimal parallel decompositions of strings a and b that give the value of distance ρ^*. This algorithm is a modification of the previously presented $BuildOPD$ algorithm. We modified the classical version of the pseudo-code to print only the most optimal decomposition (shortest), instead of printing all possible paths.

Algorithm 4: Shortest Optimal Parallel Decomposition (sOPD):
Generate shortest optimal parallel decomposition of given $x, y \in L(A)$.

`/* BackTracking Function Building Optimal Parallel Decompositions */`

Function *Build_sOPD (n,m; x,y,a,b; D)*
    if $(n = 0) and (m = 0)$ **then**
        return $(reverse(a), reverse(b))$;
    else
        if $((n > 0) and (m > 0))$ *and* $((D[n, m] = D[n-1, m-1] + dist(x_n, y_m))$ *or* $((D[n, m] = D[n-1, m-1])$ *and* $(dist(x_n, y_m) = 0)))$ **then**
            BuildOPD(n-1,m-1,a+x_n,b+y_m);
        else if $(n > 0)$ *and* $(D[n, m] = D[n-1, m] + cost_{remove})$ **then**
            BuildOPD(n-1,m,a+x_n,b+ε);
        else if $(m > 0)$ *and* $(D[n, m] = D[n, m-1] + cost_{insert})$ **then**
            BuildOPD(n,m-1,a+ε,b+y_m);

Data: $x = x_1 x_2 \dots x_n$, $y = y_1 y_2 \dots y_m$.
Parameters: costs of insertion and removal operations - $cost_{insert}$ and $cost_{remove}$ respectively.
Result: Strings representing optimal parallel decompositions.
d, D := LevenshteinDistance(x, y);
a := ”; b := ”; BuildOPD(n,m,x,y,a,b,D);

In the worst case scenario its complexity is $O(m + n)$ (this happens when we separately traverse the matrix horizontally and vertically). This result is achieved with the help of prioritizing the direction of analysis when traversing the matrix. We first look to the north-west and only afterwards to the northern and western cell values. We stop the reconstruction process once the algorithm reaches the cell at $D[0, 0]$. The reasoning behind this decision is to find the most optimal decomposition among all possible decompositions of strings a and b. The example that follows is a good illustration of this approach.

Example 3.5.1. *Let's investigate the example where $a = industry$ and $b = interest$. In this case we have $\rho^*(a, b) = 6$. The possible decompositions of strings a and b are as follows:*

industry	*in$\varepsilon\varepsilon$dustry*	*inεdεustry*	*ind$\varepsilon\varepsilon$ustry*	*inεduεstry*
interest	*interest$\varepsilon\varepsilon$*	*interest$\varepsilon\varepsilon$*	*interest$\varepsilon\varepsilon$*	*interest$\varepsilon\varepsilon$*

The first pair of parallel string decompositions is the optimal one as it has minimal string length. Another good example of two strings decomposition into their building blocks u_i, v_j, and w_j is illustrated below.

Example 3.5.2. *Consider the alphabet $\bar{A} = \{\varepsilon, X, Y, Z, W\}$ and two strings $a = XXYYWZYX$ and $b = YXXWZWXY$. For this example we obtain that $\rho^*(a, b) = 5$ as well as the following optimal decomposition:*

$$\begin{matrix} \varepsilon \\ \\ Y \end{matrix} \left(\begin{matrix} X & X \\ \\ X & X \end{matrix} \right) \begin{matrix} Y & Y \\ \\ W & Z \end{matrix} \left(\begin{matrix} W \\ \\ W \end{matrix} \right) \begin{matrix} Z \\ \\ X \end{matrix} \left(\begin{matrix} Y \\ \\ Y \end{matrix} \right) \begin{matrix} X \\ \\ \varepsilon \end{matrix}$$

Lets look at results in detection of the mutational events. We extend the parallel decompositions and present the construction of the *semiparallel decompositions*. We take into consideration the ordering $\preceq$ and the corresponding distance ρ_l^*. From this point of view, for any two strings $a, b \in L(A)$ we find the decompositions of the form $a = v_1u_1v_2u_2 \cdots v_ku_kv_{k+1}$ and $b = w_1u_1'w_2u_2' \cdots w_ku_k'w_{k+1}$, where

- u_i, u_i' are canonical substrings of the strings a and b and u_i, u_i' may be empty strings;
- v_j is a substring of a and v_j may be an empty string;
- w_j is a substring of b and w_j may be an empty string;
- $\rho_l^*(u_i, u_i') = 0$ for all $i \leq k$;

Like in the case with parallel decompositions, the semiparallel decompositions are optimal if

$$\rho_l^*(a, b) = \Sigma\{\rho(v_i, w_i) : i \leq k + 1\}.$$

This given interpretation of the metric and string decompositions can be used in the study of the minimum number of acceptable and unacceptable (when metric ρ_r^* is used) *mutational events* required to convert one sequence to another.

To illustrate the application of the semiparallel decomposition let us partition the strings from the previous example.

Example 3.5.3. *Let $a = XXYYWZYX$ and $b = YXXWZWXY$, with the alphabet $\bar{A} = \{\varepsilon, X, Y, Z, W\}$, on which we consider the classic ordering $\preceq$, meaning that $\rho_l^*(z_i, z_j) = 0$ for all $z_i, z_j \in \bar{A}$, where $z_i \preceq z_j$. This time we obtain that $\rho_l^*(a, b) = 3$, as well as the following optimal decomposition:*

$$\begin{pmatrix} X & X \\ Y & X \end{pmatrix} \begin{matrix} Y \\ X \end{matrix} \begin{pmatrix} Y \\ W \end{pmatrix} \begin{matrix} W \\ Z \end{matrix} \begin{pmatrix} Z \\ W \end{pmatrix} \begin{matrix} Y \\ X \end{matrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

For semiparallel decompositions we can define measure of similarity, penalty, and proper similarity.

Remark 3.5.1. *Our algorithms are effective for any quasi-metric on $\bar{A}$. Some authors consider the possibility to define the generalized Levenshtein metric with distinct values $\rho(a, b)$ and $\rho(b, a)$. It is necessary to require that $\rho(a, b)$ is a quasi-metric. In other cases we may obtain some confusions as will be seen from the next example.*

Example 3.5.4. *Let $A = \{a, b\}$, $\bar{A} = \{\varepsilon, a, b\}$. The following table defines the distance ρ on $\bar{A}$:*

0	0	1	ε
1	0	0	a
0	1	0	b
ε	a	b	$_y \diagdown ^x$

In this example we have $0 = \rho(a, b) + \rho(b, \varepsilon) < \rho(a, \varepsilon) = 1$ and:

1. for $u = aba$, $v = ba$ we get $\bar{\rho}(u, v) = \bar{\rho}(v, u) = 0$,

2. for $u = a$, $v = b$ we get $\bar{\rho}(u, v) = \bar{\rho}(v, u) = 0$, when $\rho(v, u) = 1$.

Example 3.5.5. *Let us examine the example from [151] in the context of the results achieved. We have strings $a = AJCJNRCKCRBP$ and $b = ABCNJROCLCRPM$ for which there are eight pairs of optimal decompositions. We present two of them, the shortest and the longest:*

$$\begin{pmatrix} A \\ A \end{pmatrix} \begin{matrix} J \\ B \end{matrix} \begin{pmatrix} C \\ C \end{pmatrix} \begin{matrix} \varepsilon \\ N \end{matrix} \begin{pmatrix} J \\ J \end{pmatrix} \begin{matrix} N & R \\ R & O \end{matrix} \begin{pmatrix} C \\ C \end{pmatrix} \begin{matrix} K \\ L \end{matrix} \begin{pmatrix} C & R \\ C & R \end{pmatrix} \begin{matrix} B & P \\ P & M \end{matrix}$$

$$\begin{pmatrix} A \\ A \end{pmatrix} \begin{matrix} J \\ B \end{matrix} \begin{pmatrix} C \\ C \end{pmatrix} \begin{matrix} J \\ \varepsilon \end{matrix} \begin{pmatrix} N \\ N \end{pmatrix} \begin{matrix} \varepsilon \\ J \end{matrix} \begin{pmatrix} R \\ R \end{pmatrix} \begin{matrix} \varepsilon \\ O \end{matrix} \begin{pmatrix} C \\ C \end{pmatrix} \begin{matrix} K \\ L \end{matrix} \begin{pmatrix} C & R \\ C & R \end{pmatrix} \begin{matrix} B \\ \varepsilon \end{matrix} \begin{pmatrix} P \\ P \end{pmatrix} \begin{matrix} \varepsilon \\ M \end{matrix}$$

For the first pair we have $\rho^* = 7$, $m = 6$, $p = 1$, and $M = 5$. For the second pair we have $\rho^* = 7$, $m = 8$, $p = 5$, and $M = 3$. Our algorithms allow us to calculate all optimal decompositions with distinct measure of similarity. Authors from [151] prefer the second pair of decomposition since it has maximal possible measure of similarity. We consider more preferable the first pair, which has the maximal proper similarity.

In fact, for any two non-empty strings there exist the parallel decompositions with maximal measure of similarity and the optimal decompositions on which measure of similarity is minimal. The following example shows that there exist some exotic non optimal parallel decompositions $a = a'_1 a'_2 \cdots a'_q$ and $b = b'_1 b'_2 \cdots b'_q$, such that for optimal decompositions $a = a_1 a_2 \cdots a_n$ and $b = b_1 b_2 \cdots b_n$ we have $m' < m$, $p' < p$ and $M' > M$.

Example 3.5.6. *Let $a = ABCDEF$ and $b = CDEFED$ be trivial non optimal decompositions of strings a, b, and $a = ABCDEF\varepsilon\varepsilon$ and $b = \varepsilon\varepsilon CDEFED$ be their optimal decompositions. Then $m' = 1$, $r' = 5$, $p' = p'_l = p'_r = 0$ and $m = 4$, $r = 4$, $p = 4$, $p_l = p_r = 2$. In this example we have that $M'_l = M'_r = M' = m' - p' = 1 - 0 = 1 > 0 = 4 - 4 = m - p = M$, $m' - r' = -4 < 0 = m - r$, $M_l = 4 - 2 = 2 > 1 = M'_l$, $M_r = 4 - 2 = 2 > 1 = M'_r$.*

Example 3.5.7. *Let $a = AAAACCC$ and $b = CCCBBBB$ be trivial optimal decompositions of strings a, b, and $a = AAAACCC\varepsilon\varepsilon\varepsilon\varepsilon$ and $b = \varepsilon\varepsilon\varepsilon\varepsilon CCCBBBB$ be their non-optimal decompositions. Then $m' = 3$, $r' = 8$, $p' = 8$ and $m = 0$, $r = 7$, $p = p_l = p_r = 0$. In this example we have that $-5 = m' - r' > m - r = -7$ and $-5 = m' - p' < m - p = 0$.*

The above examples show that Theorem 3.3.1 cannot be improved in the case of $m' < m$.

3.6. Conclusions for chapter 3

An important role plays the measures of similarity between information, how much we distinguish one sequence of information from another. Distance is one of the general methods of establishing similarity. One of the building steps of the process of computing measure of similarity is generating the parallel decompositions of a pair of strings. We presented the pseudocode of the algorithm 3 that builds all optimal parallel decompositions of two strings. Efficiency, penalty and similarity were defined for given parallel decompositions. Decompositions with minimal penalty and maximal proper similarity are of significant interest. Moreover, if we consider the problem of text editing and correction, the optimal decompositions are more favorable. Therefore, optimal decompositions are the best parallel decompositions and we may solve the string matching problems only on class of optimal decompositions.

The obtained relations between efficiency, penalty and similarity permit to formulate the following conclusions:

1. For any two non-empty strings there exist parallel decompositions with maximal measure of similarity and optimal decompositions on which measure of similarity is minimal.

2. On the class of all optimal decompositions of given two strings:

   – the maximal measure of proper similarity is attained on the optimal parallel decomposition with minimal penalties (minimal measure of similarity);

   – the minimal measure of proper similarity is attained on the optimal parallel decomposition with maximal penalties (maximal measure of similarity).

3. It was established that optimal decompositions:

   – describe the proper similarity of two strings;

   – permit to obtain long common sub-sequences;

   – permit to calculate the distance between strings;

   – permit to appreciate changeability of information over time.

4. GEOMETRICAL AND TOPOLOGICAL ASPECTS OF INFORMATION ANALYSIS

This chapter is the final chapter of this thesis, and focuses on the applicative part of the theoretical results obtained in previous chapter. More specifically, the problem of constructing the weighted means and the bisector sets of a pair of strings is solved in this chapter. Next, the analysis of the question of the convexity of the set of the weighted means is presented. The chapter concludes with the study of the image processing methods using the notions of scattered and digital spaces. One of the results of this study establishes that the Khalimsky topology is the minimal digital topology in the class of all symmetrical topologies on the discrete line $\mathbb{Z}$.

The results of the research presented in this chapter, along with the results discussed in the previous chapters fully cover the research goals stated in the first chapter. The results from this chapter are published in the articles [39, 40, 41, 42, 60, 61, 65] and can be applied in the study of various theoretical as well as practical problems.

4.1. Construction of the weighted means of a pair of strings

On the given alphabet $\bar{A} = A \cup \{\varepsilon\}$ fix a quasi-metric d with the Graev extension d_G.

Lemma 4.1.1. *Let $a, b, c \in L(A)$, $n \geq 1$ and $a' = a_1a_2...a_n$, $b' = b_1b_2...b_n$, $c' = c_1c_2...c_n$ be representations of the strings a, b, c respectively. If*

$$d_G(a, b) = \Sigma\{d(a_i, b_i) : i \leq n\} = \Sigma\{d(a_i, c_i) + d(c_i, b_i) : i \leq n\},$$

then the following assertions hold:

1. The strings $a' = a_1a_2...a_n$ and $b' = b_1b_2...b_n$ form the parallel d-optimal representations of the pair of strings a and b.

2. The strings $a' = a_1a_2...a_n$ and $c' = c_1c_2...c_n$ form the parallel d-optimal representations of the pair of strings a and c.

3. The strings $c' = c_1c_2...c_n$ and $b' = b_1b_2...b_n$ form the parallel d-optimal representations of the pair of strings c and b.

Proof: Follows from the inequality $d_G(x, y) \leq d_G(x, z) + d_G(z, y)$, for any strings $x, y, z \in L(A)$. □

We define the following sets:

$$M_{d_G}(a, b) = \{x \in L(A) : d_G(a, b) = d_G(a, x) + d_G(x, b)\}$$

and

$$M^*_{d_G}(a, b) = \{x \in L^*(A) : d_G(a, b) = d_G(a, \kappa(x)) + d_G(\kappa(x), b)\}$$

as the sets of weighted d-means of the oriented pair of strings $a, b \in L(A)$.

Assume that

$$M_{d_H}(a, b) = \{x \in L^*(A) : d_H(a, b) = d_H(a, x) + d_H(x, b)\}$$

is the set of H-weighted d-means of the oriented pair of strings $a, b \in L^*(A)$.

First, we construct equivalent representations of strings from $M_{d_G}(a, b)$ with respect to given parallel d-optimal decompositions of a and b.

Theorem 4.1.1. *Any fixed parallel d-optimal decompositions of a pair given strings $a, b \in L(A)$ generate weighted means, simultaneously with their equivalent representations, which form parallel d-optimal decompositions with the fixed representations of the given strings.*

Proof: We present the proof by construction. Let $a' = a_1a_2...a_n$ and $b' = b_1b_2...b_n$ be the fixed parallel d-optimal decompositions of the strings a and b. Denote $\bar{M}^*_{d_G}(a', b') = \{c = c_1c_2...c_n \in L^*(A) : d_H(a', c) + d_H(c, b') = d_G(a, b)\}$ and $\bar{M}_{d_G}(a', b') = \{\kappa(c) : c \in \bar{M}^*_{d_G}(a', b')\}$, where $\kappa : L^*(A) \longrightarrow L(A)$ is the operation of the identification.

We aim to construct strings of form $c' = c_1c_2...c_n$ such that for $c = \kappa(c')$ we have $d_G(a, b) = d_G(a, c) + d_G(c, b) = \Sigma\{d(a_i, c_i) + d(c_i, b_i) : i \leq n\}$.

For each $i \leq n$ we fix $c_i \in M_d(a_i, b_i) = \{x \in \bar{A} : d(a_i, x) + d(x, b_i) = d(a_i, b_i)\}$ and put $c' = c_1c_2...c_n$. Let $c = \kappa(c')$. From Lemma 4.1.1 it follows:

- the strings $a' = a_1a_2...a_n$ and $c' = c_1c_2...c_n$ form the parallel d-optimal representations of the pair of strings a and c;
- the strings $c' = c_1c_2...c_n$ and $b' = b_1b_2...b_n$ form the parallel d-optimal representations of the pair of strings c and b;
- $d_G(a, b) = d_G(a, c) + d_G(c, b)$;
- $c \in M_{d_G}(a, b)$.

The numbers $n(a', b') = |\bar{M}_{d_G}(a', b')|$ and $n^*(a', b') = |\bar{M}^*_{d_G}(a', b')|$ are estimated by the following relations:

$$n(a', b') \leq n^*(a', b') = \Pi\{|M_d(a_i, b_i)| i \leq n\},$$

$$\Pi\{|M_d(a_i, b_i)| i \leq n\} \geq 2^{|\{i \leq n : a_i \neq b_i\}|}.$$

This completes the proof of the theorem. □

In the case of discrete metric when $d_G(a, b)$ is an even number, we have the following algorithm for constructing the medians of a pair strings:

Algorithm 5: Medians of OPD of x and y:
Given $x, y \in L(\bar{A})$ construct $m \in L(\bar{A})$, s.t. $d^*(x, m) = d^*(m, y)$.

Data: $x = x_1 x_2 \dots x_n$, $y = y_1 y_2 \dots y_m$.
Result: Set M of median strings m.

$d := d^*(x, y)$;
if *d is odd* **then**
  return "distance $d^*(x, y)$ is odd, set M is an empty set."
`// Generate Optimal Parallel Decompositions of strings x,y`
$OPD(x, y) :=$ BuildOPD(x,y);
$I = \{i : 1 \leq i \leq l^*(x')\}$;
foreach $(x', y') \in OPD(x, y)$ **do**
  $I_1 = \{i : x'_i = y'_i\}$;
  $I_2 = I \setminus I_1$;
  foreach $I_3 = Choose\ (|I| - d)/2\ elements\ from\ I_2$ **do**
    $m := m_1 m_2 \dots m_{|I|}$, where $m_i = \begin{cases} x'_i, i \in I_1 \cup I_3 \\ y'_i, otherwise. \end{cases}$
  $M := M \cup \{m\}$;
return M;

Remark 4.1.1. *One can notice that the median of a pair of strings is a special case of the above theorem. In particular, if $C \subset \{1, 2, ..., n\}$, and*

$$\Sigma\{d(a_i, b_i) : i \in C\} = \Sigma\{d(a_i, b_i) : i \notin C\},$$

putting $c_i = a_i$ for $i \in C$ and $c_i = b_i$ for $i \notin C$, for $c = \kappa(c_1 c_2 ... c_n)$ we get $d_G(a, b) = 2d_g(a, c) = 2d_G(c, b)$ and c is an element of the median of a pair of strings a, b.

Further we present an important result which will be used to prove the converse of Theorem 4.1.1.

Lemma 4.1.2. *Let a, b and c be three strings for which $d_G(a, b) = d_G(a, c) + d_G(c, b)$. Then there exist $n \geq 1$ and the strings $a' = a_1 a_2 ... a_n$, $b' = b_1 b_2 ... b_n$ and $c' = c_1 c_2 ... c_n$ such that:*

1. The strings $a' = a_1 a_2 ... a_n$ and $b' = b_1 b_2 ... b_n$ form the parallel d-optimal representations of the pair of strings a and b.

2. The strings $a' = a_1 a_2 ... a_n$ and $c' = c_1 c_2 ... c_n$ form the parallel d-optimal representations of the pair of strings a and c.

3. The strings $c' = c_1 c_2 ... c_n$ and $b' = b_1 b_2 ... b_n$ form the parallel d-optimal representations of the pair of strings c and b.

4. The representation $c' = c_1 c_2 ... c_n$ of the string $c \in M_{d_G}(a, b)$ is generated by the parallel d-optimal representations $a' = a_1 a_2 ... a_n$, $b' = b_1 b_2 ... b_n$ of the pair of strings a and b.

Proof: First we examine the case when $c \sim \varepsilon$, i.e. the string c is equivalent to ε. We fix the parallel d-optimal representations $a' = a_1 a_2 ... a_n$ and $b' = b_1 b_2 ... b_n$ of the pair of strings a and b.

Then we put $c' = c_1c_2...c_n$, where $c_i = \varepsilon$ for each $i \leq n$. In this case the assertions of Lemma are proved.

Assume now that the $\kappa(c) \neq \varepsilon$. Then $l(c) = k \geq 1$. In this case we use the following algorithm:

1. Fix the parallel d-optimal representations $a^1 = u_1u_2...u_p$ and $c^1 = v_1v_2...v_p$ of the pair of strings a and c and the parallel d-optimal representations $c^2 = w_1w_2...w_m$ and $b^2 = z_1z_2...z_m$ of the pair of strings c and b.

2. We determine the sets $\{i \leq p : v_i \neq \varepsilon\} = \{i_j : j \leq k\}$ and $\{i \leq m : s_i \neq \varepsilon\} = \{s_j : j \leq k\}$, where $1 \leq i_1 < i_2 < ... < i_k \leq p$ and $1 \leq s_1 < s_2 < ... < s_k \leq m$.

3. We calculate $n_1 = max\{i_1, s_1\}, n_2 = max\{i_2 - i_1, s_2 - s_1\} + n_1, ..., n_k = max\{i_k - i_{k-1}, s_k - s_{k-1}\} + n_{k-1}, n = n_{k+1} = max\{p - i_k, m - s_k\} + n_k$.

4. Fix two monotone injection mappings $f : \{1, 2, ..., p\} \to \{1, 2, ..., n\}$ and $g : \{1, 2, ..., m\} \to \{1, 2, ..., n\}$ such that $f(i_1) = g(s_1) = n_1$ and $f(i_j) = g(s_j) = n_j$ for each $j \leq k$.

5. We construct the string $c' = c_1c_2...c_n$, where $c_{n_j} = v_{i_j} = w_{s_j}$ for each $j \leq k$ and $c_i = \varepsilon$ if $i \notin \{n_1, n_2, ..., n_k\}$.

6. Fix the representation $a' = a_1a_2...a_n$ of the string a such that $a_{n_j} = u_{i_j}$ for each $j \leq k$. We can assume that $a_{f(i)} = u_i$ for each $i \leq p$ and $a_i = \varepsilon$ for $i \notin f(\{1, 2, ..., p\})$.

7. Fix the representation $b' = b_1b_2...b_n$ of the string a such that $b_{n_j} = z_{s_j}$ for each $j \leq k$. We can assume that $b_{g(i)} = z_i$ for each $i \leq m$ and $b_i = \varepsilon$ for $i \notin g(\{1, 2, ..., m\})$.

8. The representations $a' = a_1a_2...a_n$, $b' = b_1b_2...b_n$ and $c' = c_1c_2...c_n$ are constructed.

From the above, by construction, we obtain the following:

$$d_H(a_1a_2...a_n, c_1c_2...c_n) = d_H(u_1u_2...u_p, v_1v_2...v_p) = d_G(a, c),$$

$$d_H(c_1c_2...c_n, b_1b_2...b_n) = d_H(w_1w_2...w_m, z_1z_2...z_m) = d_G(c, b),$$

$$d_G(a, b) \leq d_H(a_1a_2...a_n, b_1b_2...b_n) \leq \Sigma\{d(a_i, b_i) : i \leq n\}.$$

Also, the following equalities hold:

$$\begin{aligned}\Sigma\{d(a_i, b_i) : i \leq n\} &= \Sigma\{d(a_i, c_i) + d(c_i, b_i) : i \leq n\} \\ &= d_G(a, c) + d_G(c, b) = d_G(a, b).\end{aligned}$$

Hence $a' = a_1a_2...a_n$, $b' = b_1b_2...b_n$ and $c' = c_1c_2...c_n$ are the desired representations. The proof is complete. □

We are now ready to state the converse of Theorem 4.1.1.

Corollary 4.1.1. *Any weighted mean of a fixed pair of strings is generated by some of their optimal parallel decompositions.*

Remark 4.1.2. *Let $a, b \in L(A)$. Then from Lemma 4.1.2 it follows:*

1. Any weighted mean of a fixed pair of strings is generated by some of their optimal irreducible parallel decompositions.

2. If for any $x, y \in \bar{A}$ the set $M_d(x, y)$ of all weighted means is finite, then of the oriented pair of points $a, b \in L(A)$ the set $M_d(a, b)$ of all weighted means is finite too.

Lemma 4.1.2 is not true for arbitrary strings.

Example 4.1.1. *Let $\{0, 1\} \subset A$, where $0 \neq 1$. Consider that $d(x, x) = 0$ for any $x \in \bar{A}$ and $d(x, y) = 1$ for any distinct elements $x, y \in \bar{A}$. We say that d is the discrete metric on $\bar{A}$. Then d, d_H and d_G are metrics.*

Consider the canonical strings $a = 01$, $b = 0$ and $c = 1$. We have $d_G(a, b) = d_H(a, b) = d_G(a, c) = d_H(a, c) = d_G(c, b) = d_H(c, b) = 1$.

Fix the representations $a' = a_1a_2...a_n$, $b' = b_1b_2...b_n$ and $c' = c_1c_2...c_n$ of the strings a, b and c respectively. Assume that:

- the strings $a' = a_1a_2...a_n$ and $b' = b_1b_2...b_n$ form the parallel d-optimal representations of the pair of strings a and b.

- the strings $a' = a_1a_2...a_n$ and $c' = c_1c_2...c_n$ form the parallel d-optimal representations of the pair of strings a and c.

There exist $1 \leq i < j \leq n$ such that $a_i = 0$, $a_j = 1$ and $a_s = \varepsilon$ for $s \notin \{i, j\}$. Since $d_G(a, b) = \Sigma\{d(a_s, b_s) : s \leq n\} = 1$, we have $b_i = 0$ and $b_s = \varepsilon$ for $s \neq i$. Since $d_G(a, c) = \Sigma\{d(a_s, c_s) : s \leq n\} = 1$, we have $c_j = 1$ and and $c_s = \varepsilon$ for $s \neq j$. Thus $d_H(b_1b_2...b_n, c_1c_2...c_n) = 2 > 1 = d_G(b, c)$ and the strings $c' = c_1c_2...c_n$ and $b' = b_1b_2...b_n$ does not form the parallel d-optimal representations of the pair of strings c and b. Thus the requirement $d_G(a, b) = d_G(a, c) + d_G(c, b)$ is essential in the conditions of Lemma 4.1.2.

Example 4.1.2. *Let $\{0, 1\} \subset A$, where $0 \neq 1$. Consider that $d(x, x) = 0$ for any $x \in \bar{A}$ and $d(x, y) = 1$ for any distinct elements $x, y \in \bar{A}$. Then d, d_H and d_G are metrics.*

Let $a' = a_1a_2...a_n$ and $b' = b_1b_2w_2 \ldots b_nu_n$ be the fixed parallel d-optimal decompositions of the strings a and b. Let $N = \{i \leq n : a_i \neq b_i\}$. For any proper subset M of N we put $c_M = c_1c_2...c_n$, where $c_i = a_i$ for $i \notin M$ and $c_i = b_i$ for $i \in M$. For the improper subsets we have $c_\emptyset = a$ and $c_N = b$. As was proved in Theorem 4.1.1, $c = \kappa(c_M) \in M_{d_G}(a, b)$. We observe that $\bar{M}^*_{d_G}(a', b')$ is the set of all strings c_M, $M \subset N$, and $\bar{M}_{d_G}(a', b') = \kappa(\bar{M}^*_{d_G}(a', b'))$.

The number $n^*(a', b')$ of such strings from the set $\bar{M}^*_{d_G}(a', b')$, generated by the above method, is equal to $2^{|N|}$. We mention that the number $n(a', b')$ of the canonical strings $\bar{M}_{d_G}(a', b')$ may be $< 2^{|N|}$.

Let $a = 1$ and $b = 0000$ be the canonical representation of the given strings. We have $d_G(a, b)$ $= d_H(a, b) = 4$. For a and b we have the following parallel d-optimal decompositions $a' = 1\varepsilon\varepsilon\varepsilon$, $b' =$ 0000. These parallel decompositions generate the following eight canonical strings 1, 0, 00, 10, 000, 100, 0000, 1000. We have $\bar{M}_{d_G}(a', b') = \{1, 0, 00, 10, 000, 100, 0000, 1000\}$, $N = \{1, 2, 3, 4\}$ and $8 =$ $|C_{d_G}(a', b')| < 2^{|N|} = 2^4 = 16$. The other parallel d-optimal decompositions $a'' = \varepsilon\varepsilon\varepsilon 1$, $b'' = 0000$ of a, b present the following set of canonical strings $\bar{M}_{d_G}(a'', b'') = \{1, 0, 00, 01, 000, 001, 0000, 0001\}$ with $N = \{1, 2, 3, 4\}$. We have that $\bar{M}_{d_G}(a', b') \cap \bar{M}_{d_G}(a'', b'') = \{1, 0, 00, 000, 0000\}$.

Let $a, b \in L^*(A)$. The following remarks shows that the construction of the H-weighted d-means $c \in M_{d_H}(a, b)$ is more simple than the construction of the weighted d-means $c \in M_{d_G}(a, b)$.

Remark 4.1.3. *Let $a, b \in L(A)$. Then:*

1. If $x, y \in L^*(A)$, $x \sim y$ and $x \in M^*_{d_G}(a, b)$, then $y \in M^*_{d_G}(a, b)$.

2. If $x \in L^*(A)$, $x \sim y$, then $x \in M^*_{d_G}(a, b)$ if and only if $\kappa(x) \in M_{d_G}(a, b)$.

If $a \in L^*(A)$, $c = c_1c_2...c_n$, $n \geq 1$ and $c_i = a$ for any $i \leq n$, then we put $c = a^n$.

Remark 4.1.4. *Let $a, b, c \in L^*(A)$ and $n \geq 1$. Then $c \in M_{d_H}(a, b)$ if and only if $c \cdot \varepsilon^n \in M_{d_H}(a, b)$. The string $c = c_1c_2...c_n$ is called H-irreducible if $n = 1$ or $c_n \neq \varepsilon$. Hence are true the following two assertions:*

1. $l^*(c) \leq max\{l^*(a), l^*(b)\}$ for any H-irreducible element $c \in M_{d_H}(a, b)$.

2. If the string $c \in M_{d_H}(a, b)$ is not H-irreducible, then there exist a unique H-irreducible string $c' \in M_{d_H}(a, b)$ and a number $n = l^*(c) - l^*(c')$ such that $c = c' \cdot \varepsilon^n$.

Assume that

$$\bar{M}_{d_H}(a, b) = \{x \in M_{d_H}(a, b) : l^*(c) = max\{l^*(a), l^*(b)\}\}$$

is the set of H-weighted d-means c of the oriented pair of strings $a, b \in L^*(A)$ with $l^*(c) =$ $max\{l^*(a), l^*(b)\}$.

Remark 4.1.5. *We present below the algorithm of construction of elements from $M_{d_H}(a, b)$. From the above remark it follows that is sufficient to construct the strings $c \in M_{d_H}(a, b)$ for which $l^*(c)$ $= max\{l^*(a), l^*(b)\}$. Fix two strings $a, b \in L^*(A)$ with $p = l^*(a)$ and $l^*(b) = q$.*

1. We put $n = max\{p, q\}$.

2. We construct:

- $a' = a = a_1a_2...a_n$ and $b' = b = b_1b_2...b_n$ if $p = q$;

- $a' = a \cdot \varepsilon^{q-p} = a_1a_2...a_n$ and $b' = b\ b_1b_2...b_n$ if $p < q$;

- $a' = a = a_1a_2...a_n$ and $b' = b \cdot \varepsilon p - q = b_1b_2...b_n$ if $q < p$.

3. For each $i \leq n$ we fix $c_i \in M_d(a_i, b_i) = \{x \in \bar{A} : d(a_i, x) + d(x, b_i) = d(a_i, b_i)\}$.

4. Put $c = c_1c_2...c_n$.

5. Have $c \in \bar{M}_{d_H}(a, b)$.

By construction, we have $d_H(a, b) = d_H(a', b') = d_H(a', c) + d_H(c, b') = d_H(a, c) + d_H(c, b)$ and $c \in M_{d_H}(a, b)$.

One can observe that from $d_H(a, c) + d_H(c, b) = d_H(a, b)$ it follows that $c_i \in M_d(a_i, b_i)$ for each $i \leq n$. Therefore, any string $c \in \bar{M}_{d_H}(a, b)$ with $l^*(c) = n$ can be constructed by the above algorithm. Hence that algorithm permit to construct all strings $c \in M_{d_H}(a, b)$

The number $m^*(a, b) = |\bar{M}_{d_H}(a, b)|$ is estimated by the following relations:

$$m^*(a, b) = \Pi\{|M_d(a_i, b_i)|i \leq n\} \geq 2^{|\{i \leq n : a_i \neq b_i\}|}.$$

If d is discrete metric on $\bar{A}$ with $d(x, y) = 1$ for any pair of distinct elements $x, y \in \bar{A}$, then $M_d(x, y) = \{x, y\}$ for any $x, y \in \bar{A}$ and

$$m^*(a, b) = 2^{|\{i \leq n : a_i \neq b_i\}|} \text{ for any } a, b \in L^*(A).$$

4.2. Problem of convexity of the set of weighted means

Let (X, d) be a metric space. A subset $L \subseteq X$ is called d-convex if $M_d(a, b) \subseteq L$ for any $a, b \in L$.

On the alphabet $\bar{A} = A \cup \{\varepsilon\}$ consider the distance metric $d : d(x, x) = 0$ and $d(x, y) = 1$ for distinct $x, y \in \bar{A}$. Any subset of (A, d) is d-convex. In 2016 Professor Gh. Zbăganu informed us about the following questions:

Question 1. *Is it true that the set $M_{d_H}(a, b)$ is d_H-convex in $(L^*(A), d_H)$ for any $a, b \in L^*(A)$?*

Question 2. *Is it true that the set $M_{d_G}(a, b)$ is d_G-convex in $(L^*(A), d_G)$ for any $a, b \in L^*(A)$?*

Theorem 4.2.1. *The set $M_{d_H}(a, b)$ is d_H-convex in $(L^*(A), d_H)$ for any $a, b \in L^*(A)$.*

Proof: We can assume that $a = a_1a_2...a_n$ and $b = b_1b_2...b_n$. Then $x = x_1x_2...x_n \in M_{d_H}(a, b)$ if and only if $x_i \in \{a_i, b_i\}$ for any $i \leq n$. If $c = c_1c_2...c_n$, $f = f_1f_2...f_n$ are two strings from $M_{d_H}(a, b)$ and $x = x_1x_2...x_n \in M_{d_H}(c, f)$, then $x_i \in \{c_i, f_i\} \subseteq \{a_i, b_i\} \cup \{a_i, b_i\} = \{a_i, b_i\}$ and $x \in M_{d_H}(a, b)$. The proof is complete. □

Theorem 4.2.2. *There exists a finite alphabet A and two strings $a, b \in L(A)$ for which the set $M_{d_G}(a, b)$ is not d_G-convex.*

Proof: The proof follows from the following examples. □

For a metric space, it is easy to construct an example where d-convex property does not hold.

Example 4.2.1. *Consider a, b, c, c_1, c_2 satisfying the following relations:*

$$d(a, b) = 8, d(a, c1) = d(c1, b) = 4, d(c1, c2) = 6,$$

$$d(c1, c) = d(c, c2) = 3, d(a, c) = 6, d(c, b) = 4.$$

We have that $c_1, c_2 \in M_d(a, b)$ and $c \in M_d(c_1, c_2)$, but $c \notin M_d(a, b)$.

Although it is easy to find contradicting examples in general metric space, it is not so trivial for the case of the space of strings. The examples with weighted means of strings presented below give the answer to the above question.

Example 4.2.2. *Let $A = \{A, B, C, D, E, F, G, H, I, J, L\}$ and the strings a, b, c, c_1, c_2 defined as follows:*

$$a = CDBDHLBEBLAJ,$$
$$b = CABHDLIBFBLJG,$$
$$c_1 = CDBHDHLBEBLAJ,$$
$$c_2 = CABDHDLIBFBLAJG,$$
$$c = CDBDHDHLIBEBLAJG.$$

We compute the d_G distance values for these strings using algorithm 1:

$$d_G(a, b) = 7,\ d_G(a, c1) = 1,$$
$$d_G(c1, b) = 6,\ d_G(c1, c2) = 6,$$
$$d_G(c1, c) = d_G(c, c2) = 3,$$
$$d_G(a, c) = 4,\ d_G(c, b) = 5.$$

We have that $c_1, c_2 \in M_{d_G}(a, b)$ and $c \in M_{d_G}(c_1, c_2)$, but $c \notin M_{d_G}(a, b)$.

The previous example focuses on the weighted means of strings a and b. In the next example we present a more elegant counter-example that studies the midpoints of a pair of strings.

Example 4.2.3. *We compute the d_G distance for these strings:*

$$a = ZCBXAGB,\ b = XBDCYTGABK,$$
$$c_1 = ZCBXYTGABK,\ c_2 = XBDCYXAGB,$$
$$c = ZCBXYXAGB,$$

and get the following values:

$$d_G(a, b) = 8,\ d_G(a, c_1) = d_G(c_1, b) = 4,\ d_G(c_1, c_2) = 8,$$
$$d_G(c_1, c) = d_G(c, c_2) = 4,\ d_G(a, c) = 2,\ d_G(c, b) = 8.$$

Analogously as in previous example, we obtained that $c_1, c_2 \in M_{d_G}(a,b)$ and $c \in M_{d_G}(c_1, c_2)$, but $c \notin M_{d_G}(a,b)$.

Example 4.2.4. *Let $A = \{B, C, D, J, K, L, M, N, O, P, Q, R\}$,*

$$a = DJCJNRCKCRBP, b = DBCNJROCLCRPM,$$

$$a' = DJCNJNRCKCRBP, b' = DBCJNJROCLCRBPM,$$

$$c = DJCJNJNROCKCRBPM.$$

For the above strings, we have that:

$$d_G(a,b) = 7, d_G(a,a') = 1, d_G(a',b) = 6, d_G(a,b') = 5,$$

$$d_G(b',b) = 2, d_G(a',b') = 6, d_G(a',c) = d_G(c,b') = 3,$$

$$d_G(a,c) = 4, d_G(c,b) = 5.$$

Hence $a', b' \in M_{d_G}(a,b)$, $c \in M_{d_G}(a',b')$, but $c \notin M_{d_G}(a,b)$. Therefore, it follows that the set $M_{d_G}(a,b)$ is not convex.

In construction of strings a', b' and c we used the d_G-optimal parallel representations of pairs of strings a, b and a', b' respectively. The string a' is constructed using the following d_G-optimal parallel representations:

$$\begin{pmatrix} D \\ D \end{pmatrix} \begin{matrix} J \\ B \end{matrix} \begin{pmatrix} C \\ C \end{pmatrix} \begin{matrix} \varepsilon \\ N \end{matrix} \begin{pmatrix} J \\ J \end{pmatrix} \begin{matrix} N & R \\ R & O \end{matrix} \begin{pmatrix} C \\ C \end{pmatrix} \begin{matrix} K \\ L \end{matrix} \begin{pmatrix} C & R \\ C & R \end{pmatrix} \begin{matrix} B & P \\ P & M \end{matrix}$$

The string b' is constructed using the following d_G-optimal parallel representations:

$$\begin{pmatrix} D \\ D \end{pmatrix} \begin{matrix} J \\ B \end{matrix} \begin{pmatrix} C \\ C \end{pmatrix} \begin{matrix} J \\ \varepsilon \end{matrix} \begin{pmatrix} N \\ N \end{pmatrix} \begin{matrix} \varepsilon \\ J \end{matrix} \begin{pmatrix} R \\ R \end{pmatrix} \begin{matrix} \varepsilon \\ O \end{matrix} \begin{pmatrix} C \\ C \end{pmatrix} \begin{matrix} K \\ L \end{matrix} \begin{pmatrix} C & R \\ C & R \end{pmatrix} \begin{matrix} B \\ \varepsilon \end{matrix} \begin{pmatrix} P \\ P \end{pmatrix} \begin{matrix} \varepsilon \\ M \end{matrix}$$

The string c is constructed using the following d_G-optimal parallel representations:

$$\begin{pmatrix} D \\ D \end{pmatrix} \begin{matrix} J \\ B \end{matrix} \begin{pmatrix} C \\ C \end{pmatrix} \begin{matrix} \varepsilon \\ J \end{matrix} \begin{pmatrix} N & J \\ N & J \end{pmatrix} \begin{matrix} N \\ \varepsilon \end{matrix} \begin{pmatrix} R \\ R \end{pmatrix} \begin{matrix} \varepsilon \\ O \end{matrix} \begin{pmatrix} C \\ C \end{pmatrix} \begin{matrix} K \\ L \end{matrix} \begin{pmatrix} C & R & B & P \\ C & R & B & P \end{pmatrix} \begin{matrix} \varepsilon \\ M \end{matrix}$$

Example 4.2.5. *Let alphabet A and strings a, b, a', b', c be as in the previous example. We put $m = QQQQQQQQ$. Consider the strings amb, bma, $a'ma'$, $b'mb'$ and cmc. We obtain the* *following:*

$$d_G(amb, bma) = 14,$$

$$d_G(amb, a'ma') = d_G(a'ma', bma) = 7,$$

$$d_G(amb, b'mb') = d_G(b'mb', bma) = 7,$$

$$d_G(a'ma', b'mb') = 12,$$

$$d_G(a'ma', cmc) = d_G(cmc, b'mb') = 6,$$

$$d_G(amb, cmc) = d_G(cmc, bma) = 9.$$

Hence $a'ma'$, $b'mb'$ are from the middle of the segment $M_{d_G}(amb, bma)$, the string cmc is from the middle of the segment $M_{d_G}(a'ma', b'mb')$, but $cmc \notin M_{d_G}(amb, bma)$.

4.3. Construction of the bisector of a pair of strings

The set $B_{d_G}(a, b) = \{x \in L(A) : d_G(a, x) = d_G(x, b)\}$ is called the d_G-bisector of an oriented pair of irreducible strings $a, b \in L(A)$.

The set $B_{d_H}(a, b) = \{x \in L^*(A) : d_H(a, x) = d_H(x, b)\}$ is called the d_H-bisector of an oriented pair of strings $a, b \in L^*(A)$. Also, we denote the set $\bar{B}_{d_H}(a, b) = \{x \in B_{d_H}(a, b) : l^*(x) = max\{l^*(a), l^*(b)\}\}$, and the set $B^0_{d_H}(a, b) = \{x \in B_{d_H}(a, b) : l^*(x) \leq max\{l^*(a), l^*(b)\}\}$.

Our aim is to construct the strings from $B_{d_H}(a, b)$ and $B_{d_G}(a, b)$.

Lemma 4.3.1. $B_{d_H}(\varepsilon, \varepsilon) = B_{d_G}(\varepsilon, \varepsilon)$.

Lemma 4.3.2. *Let $a, b \in L^*(A)$, $c \in B_{d_H}(a, b)$ and $l^*(c) \geq max\{l^*(a), l^*(b)\}$. Then $c \cdot x \in B_{d_H}(a, b)$ for any $x \in B_{d_H}(\varepsilon, \varepsilon)$.*

Corollary 4.3.1. $B_{d_H}(a, b) = B^0_{d_H}(a, b) \cup \{x \cdot y : x \in \bar{B}_{d_H}(a, b), y \in B_{d_H}(\varepsilon, \varepsilon)\}$.

Lemma 4.3.3. *Let $a, b \in L^*(A)$ and $c \in B_{d_H}(a, b)$. Then $c \cdot \varepsilon \in B_{d_H}(a, b)$.*

Hence the set $B^0_{d_H}(a, b)$ is determined by the set $\bar{B}_{d_H}(a, b)$ and the set $B_{d_H}(a, b)$ is determined by the sets $\bar{B}_{d_H}(a, b)$ and $B_{d_H}(\varepsilon, \varepsilon)$. We can assume that the set $B_{d_H}(\varepsilon, \varepsilon)$ is well determined. For instance, if d is a metric, then $B_{d_H}(\varepsilon, \varepsilon) = L^*(A)$. Therefore, it is sufficient to propose methods of construction of the strings from $\bar{B}_{d_H}(a, b)$. Moreover, we can assume that $l^*(a) = l^*(b)$.

Theorem 4.3.1. *Let $a = a_1a_2...a_n$ and $b = b_1b_2...b_n$ be two strings from $L^*(A)$. There exist methods to construct elements $c = c_1c_2...c_n \in \bar{B}_{d_H}(a, b)$.*

Proof: We present the proof by construction.

Method 1.We divide the construction process of c into following steps:

Step 1. We fix two disjoint subsets P and Q of the set $\{1, 2, ..., n\}$ for which $\Sigma\{d(a_i, b_i) : i \in P\} = \Sigma\{d(b_i, a_i) : i \in Q\}$.

Step 2. For each $i \in R = \{1, 2, ..., n\} \setminus P \cup Q$ fix an element $c_i \in B_d(a_i, b_i) = \{x \in \bar{A} : d(a_i, x) = d(x, b_i)\}$.

Step 3. For each $i \in P$ we put $c_i = a_i$.

Step 4. For each $i \in Q$ we put $c_i = b_i$.

Step 5. We put $c = c_1c_2...c_n$.

We affirm that $c = c_1c_2...c_n \in \bar{B}_{d_H}(a, b)$. Indeed, $d_H(a, c) = \Sigma\{d(a_i, c_i) : i \in P\} + \Sigma\{d(a_i, c_i) : i \in Q\} + \Sigma\{d(a_i, c_i) : i \in R\} = \Sigma\{d(a_i, a_i) : i \in P\} + \Sigma\{d(a_i, b_i) : i \in Q\} + \Sigma\{d(a_i, c_i) : i \in R\} = \Sigma\{d(a_i, b_i) : i \in Q\} + \Sigma\{d(a_i, c_i) : i \in R\}$ and $d_H(c, b) = \Sigma\{d(c_i, b_i) : i \in P\} + \Sigma\{d(c_i, b_i) : i \in Q\} + \Sigma\{d(c_i, b_i) : i \in R\} = \Sigma\{d(a_i, b_i) : i \in P\} + \Sigma\{d(b_i, b_i) : i \in Q\} + \Sigma\{d(c_i, b_i) : i \in R\} = \Sigma\{d(a_i, b_i) : i \in P\} + \Sigma\{d(c_i, b_i) : i \in R\}$. Since $\Sigma\{d(a_i, b_i) : i \in P\} = \Sigma\{d(b_i, a_i) : i \in Q\}$ and $\Sigma\{d(c_i, a_i) : i \in R\} = \Sigma\{d(c_i, b_i) : i \in R\}$, we have $d_H(c, a) = d_H(c, b)$ and $c = c_1c_2...c_n \in \bar{B}_{d_H}(a, b)$.

Method 2. Method 2 is an extension of Method 1. We follow the same steps as in Method 1, with some modification :

Step 2′. For each $i \in R = \{1, 2, ..., n\} \setminus P \cup Q$ we take c_i such that $\Sigma\{d(a_i, c_i) : i \in R\} = \Sigma\{d(c_i, b_i) : i \in R\}$.

This completes the proof of the theorem. □

Remark 4.3.1. *Theorem 4.3.1 permits to propose a method of construction of some elements from the d_G-bisector $B_{d_G}(a, b) = \{x \in L(A) : d_G(a, x) = d_G(x, b)\}$:*

Step 1. *Fix the parallel decompositions $a' = a_1a_2...a_n$ and $b' = b_1b_2...b_n$ of the strings $a, b \in L(A)$.*

Step 2. *By the Method 1 construct some $c = c_1c_2...c_n \in \bar{B}_{d_H}(a, b)$.*

Step 3. *Compute $d_G(c, a)$ and $d_H(c, b)$.*

Step 4. *If $d_G(a, c) = d_H(c, b)$, then $c \in B_{d_G}(a, b)$*

Example 4.3.1. *Let $B = \{0, 1, 2, 3\} \subset A$. Consider the metric d on $\bar{A}$ for which:*

- $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for any $x, y \in \bar{A}$;

- $d(x, y) = 2$ for any distinct elements $x, y \in \bar{A}$ with $\{x, y\} \setminus B \neq \emptyset$;

- $d(0, 1) = d(0, 2) = d(1, 3) = d(2, 3) = 2$, and $d(0, 3) = d(1, 2) = 3$.

Then d, d_H and d_G are metrics.

For $a = 00$, $b = 11$ and $c = 23$ we have:

- $d_H(a, b) = d_G(a, b) = 4$;

- $d_H(a, c) = d_G(a, c) = 5$;

- $d_H(c,b) = d_G(c,b) = 5$;

- $c \in B_{d_G}(a,b) \cap \bar{B}_{d_H}(a,b)$;

- the string c is not constructible in $\bar{B}_{d_H}(a,b)$ by Method 1. String c is constructed by Method 2.

Example 4.3.2. *Let $\mathbf{c}$ be the cardinality of continuum. There exists a subset P of the interval $[1,2]$ with the following properties:*

- $|P| = \mathbf{c}$, where $\mathbf{c}$ is the power of continuum;

- if $k \geq 1$, $t_1, t_2, ..., t_k \in P$, $n_1, m_1, n_2, m_2, ..., n_k, m_k \in \mathbb{N}$ and $\Sigma\{n_i \cdot t_i : i \leq k\} = \Sigma\{m_i \cdot t_i : i \leq k\}$, then $n_i = m_i$ for each $i \leq k$.

Consider that $P = \{t_\mu : \mu \in M\}$. Let A be a non-empty set such that $2 \leq |A| \leq \mathbf{c}$. Fix on $\bar{A}$ some linear ordering $\leq$. Denote by $\{(x_\gamma, y_\gamma) : \gamma \in \Gamma\}$ the set of all ordered pairs $(x,y) \in \bar{A} \times \bar{A}$ for which $x \leq y$ and $x \neq y$. Fix an injection $\varphi : \Gamma \longrightarrow P$. We put $d(x,x) = 0$ for any $x \in \bar{A}$ and $d(x_\gamma, y_\gamma) = d(y_\gamma, x_\gamma) = \varphi(\gamma)$ for any $\gamma \in \Gamma$. Then the mapping $d : \bar{A} \times \bar{A} \longrightarrow P \cup \{0\}$ is a metric on $\bar{A}$ with the properties:

- $d(x,y) = 0$ if and only if $x = y$;

- $d(x,y) = d(y,x)$ for all $x, y \in \bar{A}$;

- if $(x,y) \neq (u,v)$ and $x \neq y$, then $d(x,y) \neq d(u,v)$;

- if $x \neq y$, then $d(x,y) \in P$.

In this case for any strings $a, b, c \in L^*(A)$ with $c \notin \{a,b\}$ we have $d_H(a,c) \neq d_H(c,b)$. Hence $B_{d_H}(a,b) = \emptyset$ for any distinct strings $a, b \in L^*(A)$. In particular, $B_{d_G}(a,b) = \emptyset$ for any distinct strings $a, b \in L(A)$.

Lemma 4.3.4. *Let d be a quasi-metric on $\bar{A}$ and a, b and c be three strings from $L(A)$. Then there exist $n \geq 1$ and the strings $a' = a_1a_2...a_n$, $b' = b_1b_2...b_n$ and $c' = c_1c_2...c_n$ such that:*

1. The strings $a' = a_1a_2...a_n$ and $b' = b_1b_2...b_n$ form the parallel representations of the pair of strings a and b.

2. The strings $a' = a_1a_2...a_n$ and $c' = c_1c_2...c_n$ form the parallel d-optimal representations of the pair of strings c and a.

3. The strings $c' = c_1c_2...c_n$ and $b' = b_1b_2...b_n$ form the parallel d-optimal representations of the pair of strings c and b.

Proof: We begin by examining the case when $c \sim e$. We fix the parallel d-optimal representations $a' = a_1a_2...a_n$ and $b' = b_1b_2...b_n$ of the pair of strings a and b. Then we put $c' = c_1c_2...c_n$, where $c_i = \varepsilon$ for each $i \leq n$. In this case the assertions of Lemma are proved.

Assume now that the $\kappa(c) \neq \varepsilon$. Then $l(c) = k \geq 1$. In this case we use the following algorithm:

1. Fix the parallel d-optimal representations $a^1 = u_1u_2...u_p$ and $c^1 = v_1v_2...v_p$ of the pair of strings a and c, and the parallel d-optimal representations $c^2 = w_1w_2...w_m$ and $b^2 = z_1z_2...z_m$ of the pair of strings c and b.

2. We determine the sets $\{i \leq p : v_i \neq \varepsilon\} = \{i_j : j \leq k\}$ and $\{i \leq m : s_i \neq \varepsilon\} = \{s_j : j \leq k\}$, where $1 \leq i_1 < i_2 < ... < i_k \leq p$ and $1 \leq s_1 < s_2 < ... < s_k \leq m$.

3. We calculate $n_1 = max\{i_1, s_1\}, n_2 = max\{i_2 - i_1, s_2 - s_1\} + n_1, ..., n_k = max\{i_k - i_{k-1}, s_k - s_{k-1}\} + n_{k-1}, n = n_{k+1} = max\{p - i_k, m - s_k\} + n = n_k$.

4. Fix two monotone injection mappings $f : \{1, 2, ..., p\} \to \{1, 2, ..., n\}$ and $g : \{1, 2, ..., m\} \to \{1, 2, ..., n\}$ such that $f(i_1) = g(s_1) = n_1$ and $f(i_j) = g(s_j) = n_j$ for each $j \leq k$.

5. $c' = c_1c_2...c_n$, where $c_{n_i} = v_{i_1} = w_{s_j}$ for each $j \leq k$ and $c_i = \varepsilon$ if $i \notin \{n_1, n_2, ..., n_k\}$.

6. Fix the representation $a' = a_1a_2...a_n$ of the string a such that $a_{n_j} = u_{i_j}$ for each $j \leq k$. We can assume that $a_{f(i)} = u_i$ for each $i \leq p$ and $a_i = \varepsilon$ for $i \notin f(\{1, 2, ..., p\}$.

7. Fix the representation $b' = b_1b_2...b_n$ of the string a such that $b_{n_j} = z_{s_j}$ for each $j \leq k$. We can assume that $b_{g(i)} = z_i$ for each $i \leq m$ and $b_i = \varepsilon$ for $i \notin g(\{1, 2, ..., m\}$.

8. The representations $a' = a_1a_2...a_n$, $b' = b_1b_2...b_n$ and $c' = c_1c_2...c_n$ are constructed.

Indeed, by construction, $d_H(a_1a_2...a_n, c_1c_2...c_n) = d_H(u_1u_2...u_p, v_1v_2...v_p) = d_G(a, c)$ and $d_H(c_1c_2...c_n, b_1b_2...b_n) = d_H(w_1w_2...w_m, z_1z_2...z_m) = d_G(c, b)$ Hence $a' = a_1a_2...a_n$, $b' = b_1b_2...b_n$ and $c' = c_1c_2...c_n$ are the desired representations. The proof is complete. □

The method presented in the above Lemma 4.3.4 of constructing strings a', b', and c' is called the *strings alignment* process of a, b, and c.

The following theorem shows that Method 1 permits to construct all points of the set $B_{d_G}(a, b)$ for the special metric d.

Theorem 4.3.2. *Let A be a non-empty set and consider on $\bar{A}$ the discrete metric d with $d(x, x) = 0$ for any $x \in \bar{A}$ and $d(x, y) = 1$ for any distinct elements $x, y \in \bar{A}$. For any $a, b \in L(A)$ and $c \in B_{d_G}(a, b)$ there exist the parallel decompositions $a' = a_1a_2...a_n$, $b' = b_1b_2...b_n$ and $c' = b_1b_2...b_n$ of the strings a, b and c, respectively, such that:*

1. $c' = c_1c_2...c_n \in \bar{B}_{d_H}(a', b')$ and the string c' is constructible by Method 1 in $\bar{B}_{d_H}(a', b')$.

2. The representations a', b', c' satisfy conditions of Lemma 4.3.4.

Proof: In this case d_H and d_G are metrics.

By virtue of Lemma 4.3.4, there exist $n \geq 1$ and the strings $a' = a_1a_2...a_n$, $b' = b_1b_2...b_n$ and $c' = c_1c_2...c_n$ such that:

1. The strings $a' = a_1a_2...a_n$ and $b' = b_1b_2...b_n$ form the parallel representations of the pair of strings a and b.

2. The strings $c' = c_1c_2...c_n$ and $a' = a_1a_2...a_n$ form the parallel d-optimal representations of the pair of strings c and a.

3. The strings $c' = c_1c_2...c_n$ and $b' = b_1b_2...b_n$ form the parallel d-optimal representations of the pair of strings c and b.

By construction, we have $d_G(a,c) = d_H(a',c') = d_H(c',b') = d_G(c,b)$. We put $Q = \{i \leq n : c_i = a_i\}$, $P = \{i \leq n : c_i = b_i\} \setminus Q$ and $R = \{1,2,...,n\} \setminus P \cup Q = \{i \leq n : c_i \notin \{a_i, b_i\}\}$. If $i \in Q \setminus P$, then $c_i = a_i = b_i$. Hence $\Sigma\{d(a_i,c_i) : i \in R\} = \Sigma\{d(c_i,b_i) : i \in R\} = |R| = r$, $d_G(a,c) = \Sigma\{d(a_i,c_i) : i \leq n\} = \Sigma\{d(a_i,c_i) : i \in P\} + \Sigma\{d(a_i,c_i) : i \in Q\} + \Sigma\{d(a_i,c_i) : i \in R\} = \Sigma\{d(a_i,c_i) : i \in P\} + \Sigma\{d(a_i,c_i) : i \in R\} = |P| + q = p + r$ and $d_G(c,b) = \Sigma\{d(c_i,b_i) : i \leq n\} = \Sigma\{d(c_i,b_i) : i \in P\} + \Sigma\{d(c_i,b_i) : i \in Q\} + \Sigma\{d(c_i,b_i) : i \in R\} = \Sigma\{d(c_i,b_i) : i \in Q\} + \Sigma\{d(c_i,b_i) : i \in R\} = p + r$. Therefore $c_i = a_i$ for $i \in P$, $c_i = b_i$ for $i \in Q$, $c_i \in B_d(a_i,b_i)$ for $i \in R$ and $\Sigma\{d(a_i,c_i) : i \in P\} = \Sigma\{d(c_i,b_i) : i \in Q\}$. The proof is complete. □

Below we present the pseudo-code of alignment algorithm of a pair of equivalent strings:

Algorithm 6: Alignment of two equivalent strings:
Given $x, y \in L(\bar{A})$ with $x \sim y$, construct $z \in L(\bar{A})$, s.t. $(z,z) \sim (x,y)$.

Data: $x = x_1x_2 \dots x_n$, $y = y_1y_2 \dots y_m$.
Result: String z.

```
i := 1, j := 1;
while (i <= n) or (j <= m) do
    if x[i] = y[j] then
        z := z + x[i];
        i := i + 1;
        j := j + 1;
    if (x[i] = ε) and (y[j] <> ε) then
        z := z + ε;
        i := i + 1;
    if (y[j] = ε) and (x[i] <> ε) then
        z := z + ε;
        j := j + 1;
return z;
```

4.4. Alexandroff spaces

For a topological space X and the points $a, b \in X$ we put $O(a) = \cap\{U \subset X : a \in U, U \textit{ is open in } X\}$ and $a \preceq b$ if and only if $b \in cl_X\{a\}$. Then $\preceq$ is an ordering on X and it is called the Alexandroff order or the Alexandroff - Birkhoff order generated by the topology of the space X [6, 34]. A binary relation $\leq$ on a space X is an order if it is reflexive, antisymmetric and transitive, i.e. for all a, b, $c \in X$, we have that:

- $a \preceq a$ (reflexivity);
- if $a \preceq b$ and $b \preceq a$, then $a = b$ (antisymmetry);
- if $a \preceq b$ and $b \preceq c$, then $a \preceq c$ (transitivity).

For a space X and point $x \in X$ we put $O(x) = \cap\{U \subset X : x \in U, U \textit{ is open in } X\}$ and $A(x) = O(x) \cup clX\{x\}$. If $y \in A(x)$, then $x \in A(y)$ and the points x, y are called adjacent points in the space X.

Any ordering $\leq$ on a set generates the topology $\mathcal{T}(\leq)$ with the base $O(x, \leq) = \{y \in X : y \leq x\} : x \in X\}$. The topological space $(X, \mathcal{T}(\leq))$ is an Alexandroff space [6, 12].

Quasi-metric [14, 147] on a set X we call a function $d : X \times X \longrightarrow R$ with the properties:

(M1): $d(x, y) \geq 0$ for all $x, y \in X$;

(M2): $d(x, y) + d(y, x) = 0$ if and only if $x = y$;

(M3): $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

If $d(x, y) = d(y, x)$ for all $x, y \in X$, then the quasi-metric d is called a metric.

A function d with the properties (M1) and (M2) is called a distance on a set X. A function d with the property (M1) is called a pseudo-distance on a set X. A function d with the properties (M1) and (M3) is called a pseudo-quasi-metric on a set X.

Let d be a pseudo-distance on X and $B(x, d, r) = \{y \in X : d(x, y) < r\}$ be the ball with the center x and radius $r > 0$. The set $U \subset X$ is called d-open if for any $x \in U$ there exists $r > 0$ such that $B(x, d, r) \subset U$. The family $T(d)$ of all d-open subsets is the topology on X generated by d. A pseudo-distance space is a sequential space, i.e. a set $B \subset X$ is closed if and only if together with any sequence it contains all its limits [87].

If d is a quasi-metric, then $T(d)$ is a T_0-topology. For any distance that statement is not true.

The pseudo-distance is an integer or a discrete pseudo-distance, if $d(x, y) \in \{0, 1, 2, ...\}$ for any $x, y \in X$ [147, 56]. If d is a discrete quasi-metric on X, then $O(a) = B(a, d, 1)$ for any point $a \in X$ and the space $(X, T(d))$ is an Alexandroff space.

If $\leq$ is an ordering on a set X, then we define two quasi-metrics d_l and d_r on X, where:

- $d_l(x, x) = d_r(x, x) = 0$ and $d_l(x, y) = d_r(y, x)$ for any $x, y \in X$;

- for $x \leq y$ and $x \neq y$ we put $d_l(x, y) = 1$, $d_l(y, x) = 0$, $d_r(x, y) = 0$, $d_r(y, x) = 1$;

- if $x \npreceq y$ and $y \npreceq x$, then $d_l(x, y) = d_r(x, y) = 1$.

In this case $d_s(x, y) = d_r(x, y) + d_l(x, y)$ is a metric. In general, a sum of quasi-metrics is also a quasi-metric, and may not be a metric.

For any points $a, b \in X$ we put $(-, a] = \{y \in X : y \leq a\}$, $[a, +) = \{y \in X : a \leq y\}$ and $[a, b] = \{y \in X : y \leq b\} \cap \{y \in X : a \leq y\}$. As in [85, 90] we say that V is an f-set if V is open and there exists a point $o_V \in V$ such that $V = [o_V, +)$. Any f-set is an ω-set. The set $L = [a, +)$ is called an ω-set and $o_L = a$.

For any point $a \in X$ we have $B(x, d_l, r) = (-, a]$ and $B(x, d_l, r) = [a, +)$ for any $r \in (0, 1]$. Obviously, $T(\leq) = T(d_r)$ is the topology induced by the ordering $\leq$.

If X is an Alexandroff space, then any set $V = O(x)$ is an f-set with $o_V = x$.

From the above, it follows:

Theorem 4.4.1. *For a topological space X the following assertions are equivalent:*

1. *X is an Alexandroff space.*

2. *Any ω-set is an f-set of X.*

3. *The topology of X is induced by some ordering.*

4. *The topology of X is generated by some integer pseudo-quasi-metric.*

4.5. Scattered and digital topologies in image processing

Any topological space X is considered to be a Kolmogorov space, i.e. a T_0-space: for any two distinct points $x, y \in X$ there exists an open subset U of X such that $U \cap \{x, y\}$ is a singleton set [87]. Denote by $cl_X F$ the closure of the set F in the space X and by $|L|$ the cardinality of the set L. Let $\omega = \{0, 1, 2, ...\}$, $\mathbb{N} = \{1, 2, ...\}$ and $\omega(n) = \{0, 1, 2, ..., n\}$ for each $n \in \omega$.

It is well known that distinct algebraic and topological structures have been introduced to accommodate the needs of information theories. In the process of studying of the continuous objects by the computer methods, they are approximated by finite objects or by digital images [1, 31, 35, 56, 79, 126, 164, 166, 183].

Digital image processing is a process which from a topological point of view may be described in the following way:

1. Fix an infinite space X (a continuous image of the original) and a property $\mathcal{P}$ of subspaces of the space X.

2. By some procedure we construct a number $n \in \omega$, a finite subset $H = \{h_i : i \in \omega(n)\} \subset \mathbb{Z}$ of levels and a finite family $\{G_i : i \in \omega(n)\}$ of open non-empty subsets of the space X with the properties:

- $G_i \cap G_k = \emptyset$ for all $0 \leq i < k \leq n$;

- for any $i \in \omega(n)$ and each $x \in G_i$ there exists an open subset $G(x)$ such that $x \in G(x) \subset G_i$ and $G(x)$ is a subset with the property $\mathcal{P}$ in X;

- the set $G = \{G_i : i \in \omega(n)\}$ is dense in X.

The set G is the $\mathcal{P}$-kernel and $X \setminus G$ is the $\mathcal{P}$-residue of the space X.

3. The intensity mapping $I_{\mathcal{P}} : X \to \omega \subseteq H$ is determined with the property: $I_{\mathcal{P}}(x) = maximal\{h_i : x \in cl_X G_i\}$ for each $x \in X$. We have $G_i \subset I_{\mathcal{P}}^{-1}(h_i)$ for each $i \in \omega(n)$.

4. On H is determined a digital topology for which the mapping $I_{\mathcal{P}}$ is continuous.

5. By some procedure we construct a finite T_0-space K and for any $x \in X$ we determine a non-empty subset $D_{\mathcal{P}}(x)$ of K such that:

- for any $c \in K$ the set $X(c) = \{x \in X : c \in D_{\mathcal{P}}(x)\}$ is closed and is called a $\mathcal{P}$-cell of X;

- for any $c \in K$ there exist $i \in \omega(n)$ and an open non-empty subset $X'(c) \subset G_i$ such that $X(c) = cl_X X'(c)$.

The family $\{X(c) : c \in K\}$ is called a $\mathcal{P}$-complex and the mapping $D_{\mathcal{P}}$ represent an approximation of X by a finite space K. Methods of constructions of the objects K, $D_{\mathcal{P}}$ and $X(c)$ are called the methods of digitalization. This procedure is known in image processing literature as "thinning", "skeletonization", "digitalization" and "segmentation" process. In the concrete situations, the $\mathcal{P}$-cells are called pixels, voxels etc. The mapping $D_{\mathcal{P}}$ can be considered as a model of a digitizer.

Typical problems arising in this context are:

- Which topological (geometrical) properties does the finite spaces H and K share with the space X?

- Is $D_{\mathcal{P}}$ or its inverse mapping continuous in some sense as a set-valued mapping?

- Classification of points and curves in the digital spaces. Study of digital invariants.

- Determine more "simple" topologically (homotopically) equivalent spaces of the space X, if the space K is too complicated.

The inverse problem of discretization and digitalization is in some sense the problem of finding a continuous model for a given finite space K.

Methods of discreteness of spaces bring us to the notions of a $\mathcal{P}$-scattered space and of a $\mathcal{P}$-decomposable space.

4.6. Algorithms and scattered spaces

In many cases it is necessary to find a procedure or an algorithm that allows us to study from a certain point of view a given space or some object. As a rule, this procedure of study can be extended to a much larger class of spaces.

Let $\mathcal{P}$ be a property of spaces. We say that the subspace Y of X has the property $\mathcal{P}$ in X if there exists a subspace Z of X with property $\mathcal{P}$ such that $Y \subset Z$. A space X is a space with local property $\mathcal{P}$ if for any point $x \in X$ there exists an open subspace U with the property $\mathcal{P}$ in X such that $x \in U$.

In [68, 82, 189, 190] were introduced the following classes of spaces.

Definition 4.6.1. *A space X is called a $\mathcal{P}$-scattered space if for any non-empty closed subspace Y of X there exists a non-empty open subset U of Y such that the subspace U has the property $\mathcal{P}$ in X.*

Definition 4.6.2. *A space X is called a $\mathcal{P}$-decomposable space if there exist an ordinal number $\alpha_0 \geq 1$ and a family $\{X_\alpha : \alpha < \alpha_0\}$ of non-empty subspaces of X such that:*

1) $X = \{X_\alpha : \alpha < \alpha_0\}$ and $X_\alpha \cap X_\beta = \emptyset$ for any $0 \leq \alpha < \beta < \alpha_0$;

2) the set $\cup\{X_\alpha : \alpha < \beta\}$ is open in X for each $\beta \leq \alpha_0$;

3) X_α is a space with local property $\mathcal{P}$ in X.
We say that $\{X_\alpha : \alpha < \alpha_0\}$ is a $\mathcal{P}$-decomposition of the space X.

If X is a $\mathcal{P}$-decomposable space, then the index of $\mathcal{P}$-decomposition $id_{\mathcal{P}}(X)$ is the minimal ordinal number α_0 for which there exists a $\mathcal{P}$-decomposition $\{X_\alpha : \alpha < \alpha_0\}$ of the space X.

In [68, 82, 189, 190] were proved the following assertions:

A1. *Any $\mathcal{P}$-scattered space is $\mathcal{P}$-decomposable. In this case $id_{\mathcal{P}}(X)$ is the index of $\mathcal{P}$-scatteredness of the space X.*

A2. *If any closed subspace of a space with property $\mathcal{P}$ is a space with the property $\mathcal{P}$, then any $\mathcal{P}$-decomposable space is $\mathcal{P}$-scattered.*

A3. *Any non-empty closed subspace of a $\mathcal{P}$-scattered space is a $\mathcal{P}$-scattered space.*

A4. *If any non-empty subspace of a space with property $\mathcal{P}$ is a space with the property $\mathcal{P}$, then any subspace of a $\mathcal{P}$-scattered space is a $\mathcal{P}$-scattered space.*

Let X be a $\mathcal{P}$-scattered space. If Y is a closed subspace of X, then $\mathcal{P}$-$kernel_X(Y) = \cup\{U \subset Y :$ *U is open in Y and has property $\mathcal{P}$* in $X\}$. Let $X_0 = \mathcal{P}$-$kernel_X(X)$ and $X_\alpha = \mathcal{P}$-$kernel_X(X \setminus \cup\{X_\beta : \beta < \alpha\})$ for each ordinal number α. Then $id_{\mathcal{P}}(X) = minimal\{\alpha : X_\alpha = \emptyset\}$.

Example 4.6.1. *Let $\mathcal{C}$ be the property of a space to be a connected space. Then any connected or locally connected space is a $\mathcal{C}$-decomposable space. A closed subspace of a connected space is not obligatory connected. The unit segment* $[0, 1]$ *in the Euclidean topology is connected and the Cantor subspace of the unit interval does not have any non-empty open connected subsets. Hence,* $[0, 1]$ *is a $\mathcal{C}$-decomposable space but is not a $\mathcal{C}$-scattered space.*

Example 4.6.2. *Let $\mathcal{S}$ be the property of a space to be a singleton space. The $\mathcal{S}$-scattered space is called a scattered space. A space X is scattered if for any non-empty subspace Y of X there exists a point $y \in Y$ such that the set $\{y\}$ is open in Y, i.e. y is an isolate point of Y. Denote by $id_s(X) = id_{\mathcal{S}}(X)$ the index of scateredness of the scattered space X. Any $\mathcal{S}$-decomposable space is scattered.*

Example 4.6.3. *Let k be the property of a space to be a compact space. A space X is k-scattered if for any non-empty closed subspace Y of X there exist a non-empty open subset U of Y and a compact subset F of Y such that $U \subset F$ [7, 188]. Any closed subspace of a k-scattered space is k-scattered. A subspace of a k-scattered space is not obligatory k-scattered. Indeed, the subspace of rationals from the unit interval* $[0, 1]$ *is not k-scattered whereas the unit interval is k-scattered. Any scattered space is k-scattered. The unit interval is k-scattered but not scattered. Any k-decomposable space is k-scattered.*

Example 4.6.4. *Let SP be the property: intersection of a countable family of open subsets is open. A space X is SP-scattered [113] if for any non-empty subspace Y of X there exists a non-empty open subset U of Y such that U is a space with the property SP. Any subspace of a SP-scattered space is*

SP-scattered. Any scattered space is SP-scattered. Any SP-decomposable space is SP-scattered. A space with the property SP is called a P-space. A point $a \in X$ of a space X is a P-point if any countable family of neighbourhoods of the point a contains a neighbourhood of the point a in X. A space X is a P-space if any point $a \in X$ is a P-point. There exists a hereditarily paracompact not scattered P-space.

Example 4.6.5. *Let FP be the property of a space to be a finite space. A space X is FP-scattered [113] if for any non-empty subspace Y of X there exists a non-empty open subset U of Y such that U is a finite set. Any subspace of a FP-scattered space is FP-scattered. A space is scattered if and only if it is FP-scattered.*

We mention the following universal theorem.

Theorem 4.6.1. *Let $\mathcal{P}$, Γ and $\mathcal{Q}$ be the properties of spaces with the following conditions:*

- *any space with property $\mathcal{P}$ has the properties $\mathcal{Q}$ and Γ;*
- *a closed subspace of the space with the property Γ is a space with the property Γ;*
- *if $Y = Z \cup S$ is a space with the property Γ, where S is a closed subspace with property $\mathcal{P}$ and Z is a subspace with local property $\mathcal{Q}$ in Y, then the space Y has the property $\mathcal{Q}$;*
- *if S and Z are open subspaces of the space Y with the property Γ, F is a subspace of Y with the property $\mathcal{P}$ and $x \in S \setminus Z \subset F$, then there exist an open subset U of Y and a subspace Φ with the property Γ such that $x \in U$, $U \subset \Phi \subset Z \cup (F \setminus Z)$ and $\Phi \setminus Z$ has the property $\mathcal{P}$.*

Then any $\mathcal{P}$-decomposable space X with the property Γ has the property $\mathcal{Q}$.

Proof. Fix a $\mathcal{P}$-decomposition $\{X_\alpha : \alpha < \alpha_0\}$ of the space X. It is sufficient to prove that X is a space with local property $\mathcal{P}$. For any point $x \in X$ we will construct an open subset Ux with the property $\mathcal{Q}$ in X such that $x \in Ux$.

If $x \in X_0$ then there exists an open subset Ux with the property $\mathcal{P}$ in X such that $x \in Ux$. Since any subspace with the property $\mathcal{P}$ in X has the property $\mathcal{Q}$ in X, then Ux has the property $\mathcal{Q}$ in X such that $x \in Ux$.

Assume that $0 < \alpha < \alpha_0$ and for any point $x \in \cup\{X_\beta : \beta < \alpha\}$ the open set Ux is constructed. Fix a point $a \in X_\alpha$. Then there exist an open subset S of X and a subset F of X with the property $\mathcal{P}$ such that $x \in S \subset \cup\{X_\beta : \beta \leq \alpha\}$ and $S \cap X_\alpha \subset F$. The set $Z = \cup\{X_\beta : \beta < \alpha\}$ is open in X, Z is a subspace with local property $\mathcal{Q}$ in X and $a \in V \setminus Z$. Hence, there exist an open subset Ua of X and a subspace Φ with the property Γ such that $a \in Ua$, $U \subset \Phi \subset Z \cup (F \setminus Z)$ and $\Phi \setminus Z$ has the property $\mathcal{P}$. Since Z and $U \cap Z$ are subspaces with local property $\mathcal{Q}$ in X, the subspace Φ has the property $\mathcal{Q}$. Hence Ua has the property $\mathcal{Q}$ in X. The proof is complete. □

Theorem 4.6.1 opens the possibility of studying $\mathcal{P}$-decomposable spaces using induction and algorithms.

In [68] Theorem 4.6.1 was proved for regular spaces and for normal T_1-spaces was introduced the invariant $dim_{\mathcal{P}}X = supremum\{dimF : F$ *has the property* $\mathcal{P}$ *and it is a closed subset of* $X\}$. For a paracompact $\mathcal{P}$-decomposable space we have $dimX = dim_{\mathcal{P}}X$ ([68], p. 19). This fact follows from Theorem 4.6.1. It is true, since any regular countable space is zero-dimensional.

Corollary 4.6.1. *If X is an SP-scattered paracompact space, then $dimX = 0$.*

In [187] it was shown that every paracompact scattered space is zero-dimensional. The authors of [113] mention: "we do not know if this conclusion holds for paracompact SP-scattered spaces". By virtue of above corollary, the response is affirmative.

Remark 4.6.1. *Let $\mathcal{P}$, $\mathcal{Q}$ and Γ be as in Theorem 4.6.1 and $\mathcal{Q}$ means that there exists a procedure that allows us to study from a concrete sense the spaces with the property $\mathcal{P}$. Then this procedure can be extended to the procedure of studying the $\mathcal{P}$-decomposable spaces with the property Γ. Indeed, assume that the properties $\mathcal{P}$, $\mathcal{Q}$ and Γ satisfy the following conditions:*

- *any space with property $\mathcal{P}$ has the property Γ;*
- *for any space Y with the property Γ and locally with the property $\mathcal{P}$ there exists an algorithm $\mathcal{Q}_1$ to study the space Y;*
- *if Z is an open non-empty subspace of the space with the property Γ, then for each point $z \in Z$ there exists an algorithm $\mathcal{Q}_2$ to construct an open subset U such that $z \in U \subset Z$ and U is a space with the property Γ;*
- *if $Y = Z \cup S$ is a space with property Γ, Z is open and locally with the property that exists an algorithm to study locally the space Z and $Y \setminus Z$ has the property $\mathcal{P}$, then there exists an algorithm $\mathcal{Q}_3$ to study the space Y;*
- *a closed subspace of the space with the property Γ is a space with the property Γ.*

Assume that X is a space with the property Γ and the $\mathcal{P}$-decomposition $\{X_\alpha : \alpha < \alpha_0\}$. Then:

1. *for any $\alpha < \alpha_0$ and any point $a \in X_\alpha$ we apply the algorithm $\mathcal{Q}_2$ of construction of an open subset Ua such that $a \in Ua \subset \cup\{X_\beta : \beta < \alpha\}$ and Ua is a space with the property Γ;*
2. *we apply the algorithm $\mathcal{Q}_3$ to study Ua;*
3. *we apply the algorithm $\mathcal{Q}_1$ to study X.*

4.7. Local finiteness and digital spaces

Let A be an Alexandroff space. On A consider the natural ordering: $a \leq b$ if and only if $b \in cl_X\{a\}$. We put $a\delta b$ if $a \leq b$ or $b \leq a$, i.e. the points a, b are comparable. The space A is a topological digital space if and only if the space A is chain-connected, i.e. for any two points $x, y \in A$ there exist a number $n = i(x, y) \in \mathbb{N}$ and a finite sequence $x_1, x_2, ..., x_n \in A$ such that $x_1 = x$, $x_n = y$ and $x_i \delta x_{i+1}$ for any $i < n$ (see [12, 122, 123, 78, 79, 4, 21, 128, 196]).

A topological space X is called:

- locally finite if each point $x \in X$ has a finite open set containing x.

- strongly locally finite if each point $x \in X$ has a finite open set and a finite closed set containing x.

A local finite space is an Alexandroff space and a scattered space. For any point $x \in A$ we put $f\text{-}dim(x, A) = |O(x)|$ and $f\text{-}dimX = sup\{f\text{-}dim(x, A) : x \in A\}$.

A local finite space is an Alexandroff space and a scattered space. For any point $x \in A$ we put $f\text{-}dim(x, A) = |O(x)|$ and $f\text{-}dimX = sup\{f\text{-}dim(x, A) : x \in A\}$.

We say that a space A is an f-bounded Alexandroff space if there is given a natural number $n \in \mathbb{N}$ such that for any point $x \in A$ there exists an open subset Wx sequence such that $x \in Wx$ and $|Wx| \leq n$, i.e. $f\text{-}dimA \leq n$.

A connected Alexandroff space is called a topological digital space.

Let $n \in \mathbb{N}$. We say that a space A is a topological n-digital space if for any two points $x, y \in A$ there exists a finite sequence $x_1, x_2, ..., x_n \in A$ such that $x_1 = x$, $x_n = y$ and $x_i \delta x_{i+1}$ for any $i < n$. A singleton space is considered topological 1-digital space. A topological space X is called a bounded digital space if A is a digital space with $f\text{-}dimX < \infty$.

A point $x \in X$ is called a maximal or a closed point of X if the set $\{x\}$ is closed in X. If $\leq$ is the ordering generated by the topology of the space X, then the maximal points coincide with the maximal points relative to the ordering $\leq$. If the set $\{x\}$ is open in X, then x is an initial or an open point of X. Denote by $Max(X)$ the set of all maximal points. If X is a weakly locally finite space, then the initial points coincide with the minimal points relative to the ordering $\leq$. If $x \in X$ is either open or closed it is called pure, otherwise it is called mixed [138]. In [85] a maximal point is called a vertex point.

Let $f : X \longrightarrow X$ be a homeomorphism and $a \in X$. Then $f\text{-}dim(f(a), X) = f\text{-}dim(x, X)$ and x is a maximal (initial) point if and only if $f(a)$ is a maximal (initial) point.

We say that a space X is s-homogeneous if for any two points $a, b \in X$ with $f\text{-}dim(a, X) = f\text{-}dim(b, X)$ there exists a homeomorphism $f : X \longrightarrow X$ for which $f(a) = b$. It is obvious that a non-discrete locally discrete space is not homogeneous.

Example 4.7.1. *Let* $X = \{1, 2, 3, ..., n, ...\}$ *be a space with the topology* $\mathcal{T} = \{\emptyset, X\} \cup \{\{1, 2, ..., n\} :$

$n \in \mathbb{N}\}$. *By construction, the point* 1 *is the unique initial point of* X *and the set of maximal points is empty. The space* X *is digital and locally finite,* f-$dim(n, X) = n$ *for any point* $n \in \mathbb{N}$.

Proposition 4.7.1. *Let* γ *be a family of open subsets of a space* X, $n \in \mathbb{N}$ *and* f-$dimX \leq n$. *If* $Max(X) \subset \cup\gamma$, *then* γ *is a cover of* X.

Proof. First, we prove the following assertions.

Claim 1. If Y *is a non-empty subspace of* X, *then* Y *is a locally finite space and* f-$dimY \leq f$-$dimX$.

This assertion is obvious.

Claim 2. If Y *is a non-empty closed subspace of* X, *then* $Y \cap Max(X) \neq \emptyset$.

Let Y be a non-empty closed subspace of the space X. For any point $a \in X$ we put $O(a) = \cap\{U \subset X : a \in U, U \text{ is open in } X\}$. The set $O(a)$ is open in X. If $a \in O(y)$ and $a \neq y$, then $O(a) \subset O(y)$. Assume that $m = maximum\{|Y \cap O(y)| : y \in Y\}$. Obviously $m \leq n$. Fix $a \in Y$ for which $|O(a) \cap Y| = m$. If $y \in Y \setminus \{a\}$, then $a \notin O(y)$. Hence $\{a\} = Y \setminus \cup\{O(y) : y \in Y \setminus \{a\}$ is a closed subset of X and $a \in Max(X)$.

Claim 3. $X = \cup\gamma$.

The set $Y = X \setminus \cup\gamma$ is closed and $Y \cap Max(X) = \emptyset$. By virtue of Claim 2, we have $Y = \emptyset$. Hence $X = \cup\gamma$. The proof is complete. □

Corollary 4.7.1. *For a locally finite space* X *the following assertions are equivalent:*

1. X *is a compact space.*

2. X *is a finite space.*

3. f-$dimX < \infty$ *and the set* $Max(X)$ *is finite.*

The Claim 2 in the proof of Proposition 4.7.1 is true for any strongly locally finite space (see [85], Theorem 8).

Let $\mathbb{I} = [0, 1]$ be the unit interval with the usual Euclidean topology generated by the metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{I}$.

A space X is arc-connected if for any ordered pair of points $a, b \in X$ there exists a continuous function $f : \mathbb{I} \to X$ such that $f(0) = a$ and $f(1) = b$. In this case we say that $f(\mathbb{I})$ is an arc with the endpoints a and b, the point a is the initial point and b is the terminal point of the arc.

Theorem 4.7.1. *Let* X *be a* T_0*-space,* $a, b \in X$, $n \in \mathbb{N}$ *and if for any two points* $x, y \in A$ *there exists a finite sequence* $x_0, x_1, x_2, ..., x_n \in X$ *such that* $x_0 = a$, $x_n = b$ *and* $x_i \delta x_{i+1}$ *for any* $i < n$. *Then there exist the set* $\{t_i \in \mathbb{I} : i \in \omega(n)\}$ *and a continuous mapping* $g : \mathbb{I} \to X$ *such that* $g(\mathbb{I}) = \{r_i \in \mathbb{I} : i \in \omega(n)\}$, $0 = t_0 < t_1 < t_2 < ... < t_n = 1$ *and* $g(t_i) = x_i$ *for any* $i \in \omega(n)$.

Proof. We apply the mathematical induction for n.

Let $a = b$ and $n = 1$. In this case $t_0 = 0$, $t_1 = 1$ and $g : \mathbb{I} \to X$ is a constant function with $g(\mathbb{I}) = \{a\} = \{b\}$.

Let $n = 1$ and $a \neq b$. We have two possible cases.

Case 1. $a \leq b$.

We put $g([0, 2^{-1})) = \{a\}$ and $g([2^{-1}, 1]) = \{b\}$. Since $\{a\}$ is an open set of the subspace $\{a, b\}$ and $\{b\}$ is closed in $\{a, b\}$, the mapping g is continuous.

Case 2. $b \leq a$.

We put $g([0, 2^{-1}]) = \{a\}$ and $g((2^{-1}, 1]) = \{b\}$. Since $\{a\}$ is a closed set of the subspace $\{a, b\}$ and $\{b\}$ is open in $\{a, b\}$, the mapping g is continuous.

For $n = 1$ the theorem is true. Assume that $m \geq 2$ and the theorem is true for any $n < m$. We put $c = x_{n-1}$. Thus, there exist the set $\{r_i \in \mathbb{I} : 1 \in \omega(n-1)\}$ and a continuous mapping $\varphi : \mathbb{I} \to X$ such that $\varphi(\mathbb{I}) = \{x_i \in \mathbb{I} : 1 \in \omega(n-1)\}$, $0 = r_0 < r_1 < r_2 < \ldots < r_{n-1} = 1$ and $g(r_i) = x_i$ for any $i \in \omega(n-1)$. We put $t_i = 2^{-1} r_i$ for any $i \in \omega(n-1)$, and put $t_n = 1$.

We have two possible cases.

Case 3. $c \leq b$.

We put $g(t) = \varphi(2t)$ for each $t \in [0, 2^{-1}]$, $g([2^{-1}, 1)) = \{c\}$ and $g(1) = b$. Since there exists an open subset U of X such that $U \cap \{c, b\} = \{c\}$, the mapping g is continuous.

Case 4. $b \leq a$.

We put $g(t) = \varphi(2t)$ for each $t \in [0, 2^{-1}]$ and $g((2^{-1}, 1]) = \{b\}$. Since there exists an open subset U of X such that $U \cap \{c, b\} = \{b\}$, the mapping g is continuous.

The proof is complete. □

Corollary 4.7.2. *Any digital space is arc-connected.*

Corollary 4.7.3. *Any connected local finite space is a digital arc-connected space.*

4.8. Discrete line and scattered spaces

The image classification problem is to find a fragmentation of an image under study into certain regions such that each region represents a class of elementary partitions (that means a set of pixels or voxtels) with the same label. The regions are separated by boundaries (see [12, 4, 196]). The article [164] is an overview of recent research of generalized topological property in the field of digital image processing.

Digital image processing is by nature a discrete process. This discrete nature causes few problems at the geometric level. At a topological level, this is however different. The notion at the base of topology, the neighborhood, is radically different from continuous spaces to discrete spaces. Algorithms based on topological information are numerous (see [128]).

Assume that the domain X of the plane $\mathbb{R}^2$ represents the image of the original Φ and that image is represented by an observed data function $I : X \to \mathbb{R}$ of the level intensity. We have $I(X) = \{c_i : 1 \leq i \leq n\}$. The function I is constructed in the following way:

- we determine for the image X the levels $\{c_i : 1 \leq i \leq n\} \subset \omega$;

- find a family $\{O_i : 1 \leq i \leq n\}$ of open subsets of X, where the $O = \cup\{O_i : 1 \leq i \leq n\}$ is dense in X, $O_i \cap O_j = \emptyset$ for $1 \leq i < j \leq n$ and O_i is the set of points of the intensity c_i;

- for any $i \in \{1, 2, ..., n\}$ and any $x \in O_i$ we put $I(x) = c_i$;

- if $x \in X \setminus \cup\{O_i : 1 \leq i \leq n\}$, then $I(x) = sup\{i : x \in cl_X O_i\}$;

- by the method of digitalization we construct a finite subset K of X which represents the original image.

In [4] it is considered that $I(x) = c_n$ for any $x \in X \setminus \cup\{O_i : 1 \leq i \leq n\}$. The process of constructing the open sets $\{O_i : 1 \leq i \leq n\}$ is called a fenestration of the topological space X (see [126]).

On $\mathbb{Z} = \{0, 1, -1, 2, -2, ..., n, -n, ...\}$ one can consider one of the following topologies:

- the left topology $\mathcal{T}_l = \{Z_{(-\infty,n)} = \{m \in \mathbb{Z} : m \leq n\} : n \in \mathbb{Z}\} \cup \{\emptyset, \mathbb{Z}\}$;

- the right topology $\mathcal{T}_r = \{Z_{(n,+\infty)} = \{m \in \mathbb{Z} : m \geq n\} : n \in \mathbb{Z}\} \cup \{\emptyset, \mathbb{Z}\}$;

- the topology of Khalimsky $\mathcal{T}_{Kh}$ with the open base $\mathcal{B}_{Kh} = \{\{2n - 1\} : n \in \mathbb{Z}\} \cup \{\{2n - 1, 2n, 2n + 1\} : n \in \mathbb{Z}\}$ [122, 123].

We mention that the function I of the domain X in the Euclidean topology in the space $(\mathbb{Z}, \mathcal{T}_l)$ is continuous.

The space $(\mathbb{Z}, \mathcal{T}_{Kh})$ is called the Khalimsky line, $(\mathbb{Z}^2, \mathcal{T}^2_{Kh})$ is called the Khalimsky plane, $(\mathbb{Z}^3, \mathcal{T}^3_{Kh})$ is called the Khalimsky space.

The Khalimsky's line, plane and space are s-homogeneous scattered locally finite non-compact spaces.

Remark 4.8.1. *1. Let D be a topological space and $g : D \to \mathbb{Z}$ be a function. For each $n \in \mathbb{Z}$ we put $O(g, n) = \cup\{U \subset X : g(U) = \{n\} : U$ is open in $X\}$. A continuous function f of D in $(\mathbb{Z}, \mathcal{T}_l)$ is a intensity level function on D provided $f(X) = \{0, 1, 2, ..., n\}$ for some $n \in \mathbb{N}$ and $O(f, i) \subset f^{-1}(i) \subset cl_D O(f, i)$ for any $i \in \{0, 1, 2, ..., n\}$.*

2. Any intensity function $f : D \to \mathbb{Z}$ determines on D the property $\mathcal{P}(f)$: a subset U of the subspace Y of the space D has the property $\mathcal{P}(f)$ if the set U is open in Y and $f(U)$ is an open singleton subset of $f(Y)$ as the subspace of the space $(\mathbb{Z}, \mathcal{T}_l)$. Relatively to this property D is a $\mathcal{P}(f)$-scattered space.

Any level intensity function on a space D generates some similarity on D. A similarity measure on a space D is a function of two variables $s : D \times D \longrightarrow R$, where $s(x, y) > 0$ and $s(x, x) - s(x, y) \geq 0$ for any $x, y \in D$ [31, 111, 112, 107, 156].

The space $\mathbb{Z} = \{0, 1, -1, 2, -2, ..., n, -n, ...\}$ is called the discrete line. The digital topologies on $\mathbb{Z}$ are important for the process of digitalization.

We say that the topology $\mathcal{T}$ on $\mathbb{Z}$ is symmetric if $(\mathbb{Z}, \mathcal{T})$ is a scattered Alexandroff space, the set $\{0\}$ is not open in $(\mathbb{Z}, \mathcal{T})$ and for any $n \in \mathbb{Z}$ the mapping $S_n : \mathbb{Z} \to \mathbb{Z}$, where $S_n(x) = 2n - x$ for each $x \in \mathbb{Z}$, is a homeomorphism. If $\mathcal{T}$ is a symmetric topology on $\mathbb{Z}$, then the translations $T_{2n} : \mathbb{Z} \to \mathbb{Z}$, where $T_{2n}(x) = 2n + x$ for all $n, x \in \mathbb{Z}$, are homeomorphisms of the space $(\mathbb{Z}, \mathcal{T})$.

Theorem 4.8.1. *For a topology $\mathcal{T}$ on $\mathbb{Z}$ the following assertions are equivalent:*

1. The topology $\mathcal{T}$ is symmetric.

2. There exists a non-empty subset $L \subset \{2n - 1 : n \in \mathbb{N}\}$ such that:

- $U_0 = \{0\} \cup L \cup \{-n : n \in L\}$ is the minimal open neighbourhood of the point 0 in the space $(\mathbb{Z}, \mathcal{T})$;

- the family $\mathcal{B}(L) = \{T_{2n}(U_0) : n \in \mathbb{Z}\} \cup \{\{2n - 1\} : n \in \mathbb{Z}\}$ is an open base of the topology $\mathcal{T}$ on $\mathbb{Z}$.

Proof. Assume that $L \subset \{2n-1 : n \in \mathbb{N}\}$ is a non-empty subset, $U_0 = \{0\} \cup L \cup \{-n : n \in L\}$ and $\mathcal{B}(L) = \{T_{2n}(U_0) : n \in \mathbb{Z}\} \cup \{\{2n - 1\} : n \in \mathbb{Z}\}$. Obviously, $\mathcal{B}(L)$ is an open base of the concrete symmetric topology $\mathcal{T}(L)$ on $\mathbb{Z}$. This fact proves the implication $2 \to 1$.

Fix a symmetric topology $\mathcal{T}$ on $\mathbb{Z}$. Let V_0 be the minimal neighbourhood of the point 0 and $M = V_0 \cap \mathbb{N}$.

Claim 1. $M \subset \{2n - 1 : n \in \mathbb{Z}\}$.

Assume that $k \geq 1$ and $2k \in M$. Then S_k is a homeomorphism and V_{2k} is a minimal open neighbourhood of the point $2k$ in the space $(\mathbb{Z}, \mathcal{T})$. By construction, we have $k \in V_0 \cap V_{2k}$ and $(\mathbb{Z}, \mathcal{T})$ is not a T_0-space, a contradiction. The Claim 1 is proved.

Claim 2. $V_0 = \{0\} \cup M \cup \{-n : n \in L\}$ *is the minimal open neighbourhood of the point* 0 *in the space* $(\mathbb{Z}, \mathcal{T})$.

This fact follows from construction and Claim 1.

Claim 3. *The set* $\{2n - 1\}$ *is open in* $(\mathbb{Z}, \mathcal{T})$ *for each* $n \in \mathbb{Z}$.

Since $(\mathbb{Z}, \mathcal{T})$ is a scattered space the set $\{a\}$ is open in $(\mathbb{Z}, \mathcal{T})$ for some $a \in \mathbb{Z}$. The points $2n$ are not isolated in the space $(\mathbb{Z}, \mathcal{T})$. Hence $a = 2k - 1$ for some $k \in \mathbb{Z}$. Since $S_{n-k}(2k - 1) = 2n + 1$ and S_{n-k} is a homeomorphism, the set $\{2n + 1\}$ is open in $(\mathbb{Z}, \mathcal{T})$ for each $n \in \mathbb{Z}$. Claim 1 is proved.

From the Claims 2 and 3 it follows that the family $\mathcal{B}(M) = \{T_{2n}(V_0) : n \in \mathbb{Z}\} \cup \{\{2n - 1\} : n \in \mathbb{Z}\}$ is an open base of the topology $\mathcal{T}$ on $\mathbb{Z}$ and $\mathcal{T} = \mathcal{T}(M)$. This fact proves the implication $1 \to 2$. The proof is complete. □

Remark 4.8.2. *1. The set of symmetric topologies on $\mathbb{Z}$ is oriented by the relation of inclusion. We have $\mathcal{T}(L) \subset \mathcal{T}(M)$ if and only if $L \subset M$. Hence, the topology $\mathcal{T}(L)$ is a minimal symmetric topology if and only if L is a singleton set.*

2. Let $m \in \{0, 1, 2, ...\}$ and $L_m = \{2m + 1\}$. Then the set $H_m = \cup\{n(2m + 1) : n \in \mathbb{Z}\}$ is an open and closed subset of the space $(\mathbb{Z}, \mathcal{T}(L_m))$. We have $\mathbb{Z} = H_m$ if and only if $m = 0$. Hence, the minimal symmetric topology $\mathcal{T}(L_m)$ is a digital topology if and only if $m = 0$.

3. The topology of Khalimsky $\mathcal{T}_{Kh}$ with the open base $\mathcal{B}_{Kh} = \{\{2n - 1\} : n \in \mathbb{Z}\} \cup \{\{2n - 1, 2n, 2n + 1\} : n \in \mathbb{Z}\}$ is of the form $\mathcal{T}(L)$ for $L = \{1\} = L_0$. Therefore the topology of Khalimsky is the unique minimal digital symmetric topology on the discrete line $\mathbb{Z}$.

4.9. Conclusions for chapter 4

In Chapter 4 the concept of the parallel decompositions of a given pair of strings developed for solving geometrical and topological problems associated with distinct problems of information analysis. In particular:

1. Any pair of optimal parallel decompositions of the given two strings allows us to construct some set of the weighted means and/or bisectors of these strings. It was also shown that any weighted mean is generated by some optimal parallel decompositions. These moments have solved negatively the problem of convexity of the two-string segment. Therefore, parallel decompositions:

    - permit the calculation of the median of two strings;
    - permit the calculation of the weighted means of two strings;
    - permit to solve the problem of convexity of the weighted means of two strings.
    - permit to analyze the properties of a bisector of two strings;

2. Were proposed the algorithms of image processing using the notions of scattered and digital spaces. In the class of Alexandroff topologies on the discrete line were specified the symmetrical topologies. It was established that the Khalimsky topology is the minimal digital topology in the class of all symmetrical topologies on the discrete line $\mathbb{Z}$.

GENERAL CONCLUSIONS AND RECOMMENDATIONS

The research carried out within the Ph.D. thesis "Distances on Free Monoids and Their Applications in Theory of Information" fully corresponds to the goals and the objectives set out in the introduction chapter.

The study of the results obtained permit to highlight the following general results:

1. It was established that for any non-Burnside quasivariety $\mathcal{V}$ and any quasi-metric ρ on a set X with basepoint p_X on free monoid $F^a(X, \mathcal{V})$ there exists a unique stable quasi-metric $\hat{\rho}$ with the properties:
   (a) $\rho(x, y) = \hat{\rho}(x, y)$ for all $x, y \in X$.
   (b) If d is an invariant quasi-metric on $F^a(X, \mathcal{V})$ and $d(x, y) \leq \rho(x, y)$ for all $x, y \in X$, then $d(x, y) \leq \hat{\rho}(x, y)$ for all $x, y \in F^a(X, \mathcal{V})$.
   (c) If ρ is a metric, then $\hat{\rho}$ is a metric as well.
   (d) If $Y \subseteq X$, $d = \rho|Y$ and $\hat{d}$ is the maximal invariant extension of d on $F^a(Y, \mathcal{V})$, then $F^a(Y, \mathcal{V}) \subseteq F^a(X, \mathcal{V})$ and $\hat{d} = \hat{\rho}|F^a(Y, \mathcal{V})$.
   (e) For any quasi-metric ρ on X and any points $a, b \in F^a(X, \mathcal{V})$ there exists $n \in N$ and representations $a = a_1a_2...a_n$, $b = b_1b_2...b_n$ such that $a_1, b_1, a_2, b_2, ..., a_n, b_n \in X$ and $\hat{\rho}(a, b) = \sum\{\rho(a_i, b_i) : i \leq n\}$. [62]

2. The method of extension of quasi-metrics on free monoids in the complete non-Burnside quasivariety of topological monoids permit: to construct distinct admissible topologies of $F^a(X, \mathcal{V})$ for any T_0-space X, to prove that the free topological monoid $F^a(X, \mathcal{V})$ exists for any space X, to establish that the free topological monoid $F(X, \mathcal{V})$ is abstract free, i.e. is canonically isomorphic with the abstract free monoid $F^a(X, \mathcal{V})$ [44, 57, 62].

   This fact solves problems posed by A. I. Maltsev for free universal topological algebras [133]. Similar results were obtained for quasivarieties of semi-topological monoids as well [62].

3. It was proved that if $\mathcal{V}$ is a complete non-Burnside quasivariety of topological monoids, then X is an Alexandroff space if and only if $F(X, \mathcal{V})$ is an Alexandroff space, and X is a digital space if and only if $F(X, \mathcal{V})$ is a digital space [61].

   We mention that conclusions 1, 2 and 3 do not hold for complete Burnside quasivarieties.

4. Based on distance extension methods, the notions of parallel decompositions and the measure of similarity were introduced in the space of strings [63]. Theorem 3.3.1 was presented, which describes the relationships between measure of similarity, penalty and optimality of parallel decompositions [56, 58, 59].

5. Different interesting relations between Hamming, Levenshtein and Graev distances were established under the class of $L(A)$ [36, 37, 38, 55].

6. It was proved that on the class of all optimal decompositions of given two strings the maximal measure of proper similarity is attained on the optimal parallel decomposition with minimal penalties (minimal measure of similarity), and the minimal measure of proper similarity is attained on the optimal parallel decomposition with maximal penalties (maximal measure of similarity) [43, 63].
7. Algorithms were proposed for constructing the elements of the sets of weighted means $M_{d_G}(a, b)$ and bisector $B_{d_G}(a, b)$ of a given pair of strings a and b [39, 64]. It was illustrated how to use optimal parallel decompositions to generate elements of $M_{d_G}(a, b)$, $B_{d_G}(a, b)$, and the set of midpoints between a and b [42, 41, 64].
8. It was proved that any weighted mean of a pair of strings is generated by some of their optimal parallel decompositions [64]. It was also proved that the set $M_{d_G}(a, b)$ is not convex [40].
9. Algorithms for digital image processing were elaborated using the properties of scattered and digital topologies, and it was established that the Khalimsky topology is the minimal digital topology in the class of all symmetrical topologies on the discrete line $\mathbb{Z}$ [60, 61, 65].

Advantages and value of thesis results. The proposed elaborations have a significant scientific value due to their high degree of novelty and originality. The scientific results in this thesis have a theoretical and applicative value in domains of algebra, topology and theoretical computer science. For example, the methods of extensions of pseudo-quasimetrics that can be used for construction of special topologies on free monoids. The methods of parallel decompositions, measure of similarity, efficiency and penalty can be applied in text analysis problems.

Recommendations. The results obtained can be used in various fields and may have practical applications in algebra and theory of information. Based on the above conclusions, we recommended the following:

- there is a special interest in investigating quasimetrics on the space of free monoids, as extensions of quasimetrics with particular properties on an alphabet. For instance, as it was proved, quasimetric are strictly invariant on rigid quasivarieties. This is usual for groups, but it is very rare for semigroups and monoids;
- the results research can be continued both from algebraic and applicative points of view. Researching metrics on monoids is of particular interest;
- the results obtained with optimal parallel decompositions can be used in the domain of sequences alignment;
- the new algorithm proposed for weighted means construction can be more effective because it takes into consideration the empty symbol, and generates more elements of the M_{d_G} set than the classical algorithms. This fact, in its turn, can be useful in the context of information

communication through the channel with noise, or text editing/correction software, where the loss of information is admissible;

- algorithms for generating weighted means and bisectors of strings can be applied in the domain of data analysis and clustering algorithms. For instance, the geometrical centroid of a set of elements can be calculated as the intersection of the bisectors of elements.
- further research can be continued with the study of algorithms and properties of optimal parallel decompositions of three and more strings;
- thesis contents can serve as a platform for university facultative and optional courses.

References

[1] Abramsky S., Jung A. *Domain Theory in Handbook of Logic in Computer Science*, vol. 3, edited by S. Abramsky, Dov. Gabbay, and T.S.E. Maibaum eds., Oxford University Press, 1994, p. 1–168. ISBN 0-19-853762-X.

[2] Adamek J., Trnkova V. *Automata and Algebras in Categories*. Kluwer Academic Publishers:Dordrecht and Prague, 1990. ISBN 978-0-7923-0010-6.

[3] Adami C. *Artificial Life*. Cambridge, MA: MIT Press, 1998. ISBN 978-1-4612-7231-1.

[4] Al-Hajri M., Belaid K., Belaid L.J. *Scattered spaces, compactifications and an application to image classification problem*, Tatra Mountains Math. Publ., vol. 66, 2016, p. 1–12. ISSN 1210-3195.

[5] Aleshin S. V. *Automata in algebra*. Fundam. Prikl. Mat., vol.15, 2009, no. 3, p. 23–32. ISSN 1573-8795.

[6] Alexandroff P. S. *Diskrete Raume*. Mat. Sb. (N.S.) (in German). vol. 2, 1937, p. 501–518.

[7] Alexandroff P. S., Proskuryakov I. V. *On reducible sets*, Izvestia Akad. Nauk SSSR, Matematika, vol. 5, 1941, nr. 3, p. 217–224.

[8] Alexandroff P., Urysohn P. *Une condition nécésare et suffisante pour qu'une classe (L) soit une classe (B)*, C. R. Acad. Paris, nr. 177, 1923, p. 1274–1276.

[9] Alexandroff P, Urysohn P. *Mémoire sur les espaces topologiques compacts*, Verh. Akad. Wetensch. Afd. Naturk. Sect. I, Amsterdam, vol.14, 1929, p. 1–96.

[10] Alexandru A., Ciobanu G. *Fuzzy sets within Finitely Supported Mathematics*, Fuzzy Sets and Systems, vol. 339, 2018, p. 119–133. ISSN 0165-0114.

[11] Arbib M. A. *The Algebraic Theory of Machines, Languages, and Semigroups*, Academic Press, New York, 1968.

[12] Arenas F.G. *Alexandroff spaces*. Acta Math. Univ. Comenianae, nr. 68, vol. 1, 1999, p. 17–25. ISSN 0862-9544.

[13] Arhangelskii A., Tkachenko M. G. *Topological groups and related structures*. Atlantis Press, Paris, 2008. ISBN 978-94-91216-35-0.

[14] Arhangelskii A. V. *Mappings and spaces*. Uspehi Matem. Nauk 21 (1966), vyp. 4, p. 133–184 (English translation: Russian Math. Surveys, vol. 21, 1966, no. 4, p. 115–162).

[15] Arhangelskii A.V., Choban M.M., Kenderov P.S. *Topological games and topologies on groups*, Math. Maced., nr. 8, 2010, p. 1–19. ISSN 1409 9721.

[16] Arhangelskii A.V., Choban M.M., Kenderov P.S. *Topological games and continuity of group operations*, Topology and its Appl., nr. 157, 2010, p. 2542–2552. ISSN 0166-8641.

[17] Arnautov V. I., Ermakova G. N. *Properties of finite unrefinable chains of ring topologies for nilpotent rings*, Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, nr.1 (86), 2018, p. 67–75. ISSN 1024-7696.

[18] Arnautov V. I., Ermakova G. N. *Unrefinable chains when taking the infimum in the lattice of ring topologies for a nilpotent ring*, Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, nr.2 (84), 2017, p. 71–76. ISSN 1024-7696.

[19] Arnautov V. I., Ermakova G. N. *Lattice of all topologies of countable module over countable rings*, Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, nr.2 (81), 2016, p. 63–70. ISSN 1024-7696.

[20] Aronszajn N., Panitchpakdi P. *Extensions of uniformly continuous transformations and hyperconvex metric spaces*. Pacific Journal of Mathematics. nr. 6, 1956, p. 405–439.

[21] Aubert G. Kornprobst P. *Mathematical problems in image processing*, Series: Appl. Math. Sci., vol. 147, Springer-Verlag, New York, 2002. ISBN 978-0-387-44588-5.

[22] Ayala J. *Integrated Circuit and System Design*. Springer, 2012. 255 p. ISBN 978-3-642-24154-3.

[23] Babenko I. K., Bogatyi S. A. *On topologies on the group* $(Z_p)^N$. Topology and its Applications, vol. 221, 2017, p. 638–646. ISSN 0166-8641

[24] Baez D.J., *The octonions*, Bull. Amer. Math. Soc. nr. 39, 2002, p. 145–205.

[25] Bakhtin I. A., *The contraction mapping principle in almost metric spaces*, Funct. Anal., Ulianovskii Gosud. Pedag. Inst. nr. 30 (1989), p. 26–37.

[26] Banach S. *Théorie des opérations linéaires, Warsaw*, 1932. 254 p.

[27] Barakhnin V., Nekhaeva V., Fedotov A. *Similarity Determination for Clustering Textual Documents*. In: Wolff K.E., Palchunov D.E., Zagoruiko N.G., Andelfinger U. (eds) Knowledge Processing and Data Analysis. KPP 2007, KONT 2007. Lecture Notes in Computer Science, vol 6581. Springer, Berlin, Heidelberg. ISBN 978-3-642-22139-2.

[28] Berinde V., Choban M. M. *Remarks on some completeness conditions involved in several common fixed point theorems*, Creat. Math. Inform. vol. 19, 2010, nr. 1, p. 1–10.

[29] Berinde V., Choban M. M. *Generalized distances and their associate metrics.* Impact on fixed point theory, Creat. Math. Inform. vol. 22, 2013, nr. 1, p. 23–32. ISSN 1584 – 286X.

[30] Belousov V.D. *Foundations of the theory of quasigroups and loops* (in Russian), Izdatyelstvo, Moscow, 1967.

[31] Bentley J.L., Weide B.W., Yao A.C. *Optimal expected-time algorithms for closest point problems*, ACM Trans. Math. Software vol. 6, 1980, p. 563–580. ISSN 0098-3500.

[32] Birkhoff G. *Lattice Theory*, 3rd ed., American Mathematical Society, Providence, R. L, 1967.

[33] Birkhoff G. *On the Structure of Abstract Algebras*, Proc. Cambridge Philos. Soc. vol. 31, 1935, p. 433–454.

[34] Birkhoff G. *A note on topological groups*, Compositio Math., vol. 3, 1936, p. 427–430.

[35] Blanck J. *Domain representations of topological spaces*, Theoretical Computer Science, vol. 247, 2000 , p. 229–255. ISSN 0304-3975.

[36] **Budanaev I. A. *On Hamming Type Distance Functions*, CAIM 2016: The 24^{th} Conference on Applied and Industrial Mathematics, Craiova, Romania, September 15-18, 2016, Book of Abstracts, Editura SITECH, Craiova, p. 15–16. ISSN 2537 - 2688.**

[37] **Budanaev I. A. *On Hamming Type Distance Functions*, Romai Journal 2016, vol. 12, nr. 2, p. 25–32. ISSN 1841-5512.**

[38] **Budanaev I. A. *About Distance Functions of Hamming Type*. In: Tendinţe Contemporane ale Dezvoltării Ştiinţei: Viziuni ale Tinerilor Cercetători. Materialele Conferinţei Ştiinţifice a Doctoranzilor, Universitatea AŞM, Ediţia a V-a. Chisinău, 2016, p. 297–301.**

[39] **Budanaev I. A. *About the Construction of the Bisector of Two Strings*, Romai Journal, vol.13 n.2, 2017, p. 1 – 11. ISSN 1841-5512.**

[40] **Budanaev I. A. *About Non-Convexity of the Weighted Mean of a Pair of Strings*, International Conference "Contemporary Trends in Science Development: Visions of Young Researchers", 7th Edition, Chisinau, 15 June 2018, Proceedings vol. 1, p. 6–10.**

[41] **Budanaev I. A. *On the Midset of Pairs of Strings*, International Conference on Mathematics, Informatics and Information Technologies Dedicated to the Illustrious Scientist Valentin Belousov, MITI 2018, Communications, p. 134–135. ISBN 978-9975-3214-7-1.**

[42] **Budanaev I. A.** ***On the Bisector of a Pair of Strings*****, Conference of Mathematical Society of Moldova: Proceedings CMSM4 2017, June 25 – July 2, 2017, Chisinau, Republic of Moldova, p. 475–478, 2017, ISBN: 978-9975-71-915-5.**

[43] **Budanaev I. A.** ***Parallel Decompositions and The Weighted Mean of a Pair of Strings*****, International Conference "Contemporary Trends in Science Development: Visions of Young Researchers", 6th Edition, Chisinau, 15 June 2017, Proceedings p. 7–11.**

[44] **Budanaev I. A., Choban M. M.** ***Invariant Distances on Free Semigroups and Their Applications*****. Rezumatele Comunicărilor, A XX-a Conferinţă Anuală a Societăţii de Ştiinţe Matematice din România, Baia Mare, 19-22 mai 2016, p. 17–18.**

[45] Burago D., Burago Yu., Ivanov S. *A Course in Metric Geometry*. Graduate Studies in Mathematics. Vol 33. American Mathematical Society. 2001. ISBN 978-0-8218-2129-9.

[46] Burtseva L., Cojocaru S., Gaindric C. *Images and rules based decision support system in ultrasound examination of abdominal zone*, Memoirs of the Scientific Sections of the Romanian Academy. Computer Science, SERIES IV, Tome XXXIV, 2011, p. 173–184.

[47] Camerzan I. *Reţele Petri*. Teorie şi Practică, Chişinău, Elena-V.I. SRL, 2010.

[48] Chepoi V. *A TX approach to some results on cuts and metrics*. t, p. 453–470.

[49] Chittenden E. W. *On the equivalence of écart and voisinage*, Trans. Amer. Math. Soc. vol. 18, 1917, p. 161–166.

[50] Choban M. M. *Algebre Universale Topologice,* Editura Universităţii din Oradea, Oradea 1999.

[51] Choban M. M. *On the theory of topological algebraic systems*, Trans. Moscow Math. Soc., vol. 48, 1986, p. 115–159 (Russian original: Trudy Mosk. Matem. Ob-va, vol. 48, 1985, p. 106–149).

[52] Choban M. M. *The theory of stable metrics*, Math. Balkanica, vol. 2, 1988, p. 357–373. ISSN 0205-3217.

[53] Choban M. M. *General conditions of the existence of free algebras*, Acta Comment. Univ. Tartuensis, vol. 836, 1989, p. 157–171.

[54] Choban M. M. *Some topics in topological algebra*, Topol. Appl., vol. 54, 1993, nr. 54, p. 183–202.

[55] **Choban M. M., Budanaev I. A.** ***Distances on Monoids of Strings and Their Applications*****, Proceedings of the Conference on Mathematical Foundations of Informatics MFOI2016, July 25-29, 2016, Chisinau, Republic of Moldova, 2016, p. 144–159. ISBN 978-9975-4237-4-8.**

[56] **Choban M. M., Budanaev I.** ***About Applications of Distances on Monoids of Strings*****, Computer Science Journal of Moldova, 2016, nr. 3, p. 335–356. ISSN 1561-4042.**

[57] **Choban M. M., Budanaev I. A.** ***On the Theory of Free Topological Monoids and its Applications*****, International Conference "Mathematics and Information Technologies: Research and Education" (MITRE-2016). Abstracts. Chisinau, June 23-26, 2016, p. 21–22. ISBN 978-9975-71-794-6.**

[58] **Choban M. M., Budanaev I. A.** ***Parallel Decompositions of Pairs of Strings and Their Applications*****, Presented in plenary at CAIM 2017: The 25^{th} Conference on Applied and Industrial Mathematics, Iaşi, Romania, September 14-17, 2017, Book of Abstracts, p. 64–65.**

[59] **Choban M. M, Budanaev I. A.** ***Measures of Similarity on Monoids of Strings*****, Conference on Mathematical Foundations of Informatics: Proceedings MFOI 2017, November 9-11, 2017, Chisinau, Republic of Moldova, Chisinau, Institute of Mathematics and Computer Science, p. 51–58, 2017. ISBN 978-9975-4237-6-2**

[60] **Choban M., Budanaev I.** ***About applications of topological structures in computer sciences and communications*****, Acta Et Commentationes, Ştiinţe Exacte şi ale Naturii, Revistă Ştiinţifică, nr.2 (4), 2017, p. 45–59. ISSN 2537-6284.**

[61] **Choban M. M., Budanaev I. A.** ***Scattered and Digital Topologies in Image Processing*****, Proceedings of the Conference on Mathematical Foundations of Informatics, MFOI 2018, July 2-6, 2018, Chisinau, Republic of Moldova, Chisinau, p. 21–40, 2018, ISBN: 978–9975–4237–7–9.**

[62] **Choban M. M., Budanaev I. A.** ***Distances on Free Semigroups and Their Applications*****, Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica, nr. 1, vol. 86, 2018, p. 92 – 119. ISSN 0236-3089.**

[63] **Choban M. M., Budanaev I. A.** ***Efficiency and Penalty Factors on Monoids of Strings*****, Computer Science Journal of Moldova, vol.26, nr.2 (77), 2018, p. 99 – 114. ISSN 1561-4042.**

[64] **Choban M. M., Budanaev I. A.** ***About the Construction of the Weighted Means of a Pair of Strings*****, Romai Journal, vol.14, nr.1, 2018, p. 73 – 87. ISSN 1841-5512.**

[65] **Choban M. M., Budanaev I. A.** ***Scattered and Digital Topologies in Information Sciences*****, The 26th Conference on Applied and Industrial Mathematics, CAIM 2018, Septmeber 20–23, 2018, Chisinau, Book of Abstracts, p. 9 – 11.**

[66] Choban M. M., Chiriac L. L. *Selected problems and results of topological algebra*, ROMAI J., vol. 9, 2013, nr. 1, p. 1–25. ISSN 1841-5512.

[67] Choban M. M., Chiriac L. L. *On free groups in classes of groups with topologies*, Bul. Acad. Ştiinţe Repub. Moldova, Matematica, 2013, nr 2-3, p. 61–79. ISSN: 0236-3089.

[68] Choban M.M., Dodon N.C. *Theory of P-dispersed spaces*, Chişinău, Ştiința, 1979.

[69] Choban M.M., Kenderov P.S., Moors W.B. *Eberlein theorem and norm continuity of pointwise continuous mappings into function spaces*, Topology and its Applications, vol. 169, 1 June 2014, p. 108–119. ISSN 0166-8641.

[70] Choban M. M., Pavel D. I. *Almost periodicity of functions on universal algebras.* Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, vol. 2, nr. 81, 2016, p. 45–62. ISSN 0236-3089.

[71] Chiriac L. L. *Topological Algebraic Systems*, Editura Ştiinţa, Chişinău, 2009.

[72] Ciobanu G., Todoran E. N. *Denotational semantics of membrane systems by using complete metric spaces*, Theor. Comput. Sci., vol. 701, 2017, p. 85–108. ISSN 0304-3975.

[73] Ciobanu G., Vaideanu C. *An efficient method to factorize fuzzy attribute-oriented concept lattices*, Fuzzy Sets and Systems, vol. 317, 2017, p. 121–132. ISSN 0165-0114.

[74] Clifford A.H., Preston G.B. *The algebraic theory of semigroups*, Math Surveys No. 7, Amer. Math. Soc. Providence, vol. 1, 1961, vol. 2, 1967. ISSN 1088-6826.

[75] Cojocaru S., Petic M., Titchiev I., *Adapting Tools for Text Monitoring and for Scenario Analysis Related to the Field of Social Disasters*, International Journal of Computer, Electrical, Automation, Control and Information Engineering, vol. 10, nr. 10, 2016, p. 1601–1607. ISSN 1740-7524.

[76] Conway J. H. *Regular algebra and finite machines*. Chapman and Hall Mathematics Series. London: Chapman & Hall, 1971. ISBN 9780486485836.

[77] Cormen T. H., Leiserson C. E., Rivest R. L., Stein C. *Introduction to Algorithms*. (3rd ed.), MIT Press and McGraw-Hill, 2009. 1312 p. ISBN 978-0262033848.

[78] Davey B. A., Priestley H. A. *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 1990. ISBN 978-0521784511.

[79] Denning D.E. *A lattice model of secure information flow*, Communications of the ACM. vol. 19 , nr. 5, 1976, p. 236–243.

[80] Deza E., Deza, M. M. *Dictionary of Distances*, Elsevier, Amsterdam, 2006. ISBN 9780444520876.

[81] Deza M. M., Deza, E. *Encyclopedia of Distances*, Springer, Berlin, 2014. ISBN 978-3-642-00234-2.

[82] Dodon N.C., Choban M.M. *Mappings of P-scattered spaces*, C.R. Acad. Bulgare vol. 28, nr. 1975, nr. 4, p. 441–443.

[83] Droste M., Göbel R. *Non-deterministic information systems and their domains*, Theoret. Comp. Sci., vol. 75, 1990, p. 289–309. ISSN: 0304-3975.

[84] Dumitrascu S. S., Choban M. M. *On free topological algebras with a continuous signature*. In: Algebraic and topological systems. Matem. Issledovania, vol. 65, 1982, p. 27–54.

[85] Eckhardt U., Latecki L. *Digital Topology*, Hamburger Beiträage zur Angewandten Mathematik Reihe A, Preprint vol. 89, October 1994, 31 p.

[86] Eilenberg S., Wright J. B. *Automata in General Algebras*, Information and Control, vol. 11, 1967, p. 452–470.

[87] Engelking, R., *General Topology*, PWN, Warszawa, 1977, 540 p.

[88] Ershov Yu. L. *Theory of enumerations*, Moskva, Nauka, 1977.

[89] Ershov Yu. L. *Theory of A-spaces*, Algebra i Logika, Novosibirsk SO AN, vol. 12, 1973, nr. 4, p. 269-416.

[90] Ershov Yu. L. *Definability and Computability*, Springer, 1996. ISBN 978-0-306-11039-9.

[91] Ershov Yu. L. *Model C of partial continuous functionals*, in: R.O. Gandy, J.M.E. Hyland (Eds.), Logic Colloquium 76, Studies in Logic and Foundations in Mathematics, vol. 87, North-Holland, Amsterdam, 1977, p. 455–467.

[92] Espinola R., Khamsi M. A. *Introduction to hyperconvex spaces*. In Kirk, W. A.; Sims B. (Eds.). Handbook of Metric Fixed Point Theory. Dordrecht: Kluwer Academic Publishers. 2001. ISBN 978-90-481-5733-4.

[93] Frechet M.M. *Sur quelques points du calcul fonctionnel*, Rendic. Circ. Mat. Palermo, vol. 22, 1906, p. 1–74.

[94] Frink A. H. *Distance functions and the metrization problem*, Bull. Amer. Math. Soc., vol. 43, 1937, p. 133–142.

[95] Gaindric C., Cojocaru S., Popcova O., Puiu S., Secrieru Iu. *Emergency-SonaRes: A System for Ultrasound Diagnostics Support in Extreme Cases.* In: H.-N.L. Teodorescu et al. (eds.), Improving Disaster Resilience and Mitigation - IT Means and Tools, NATO Science for Peace and Security Series C: Environmental Security, Part I, Fundamentals and Modeling, Chapter 18, 2014, p. 283–292. ISBN 978-94-017-9136-6.

[96] Glushkov V. M. *Abstract Theory of Automata*, Uspehi Matematiceskih Nauk 16:5 (1961), 3-62, (English translation:Russian Mathematical Surveys, 1961, 16:5, 1-53; Romanian translation: Analele Romano-Sovetice, Seria Matematica-Fizica, vol. 2, nr. 41, 1962, p. 3-61).

[97] Gratzer G. *Universal algebra*, 2nd ed., Springer, 1979. ISBN 978-0-387-77487-9.

[98] Graev M.I. *Free topological groups*, Topology and Topological Algebra, Transl. Ser. 1, Vol. 8 (Amer. Math. Sot., 1962) 305-364; Russian original in: Izv. Akad. Nauk SSSR Ser. Mat. 12, 1948, p. 279–323.

[99] Graev M. I. *Theory of topological groups. I. Norms and metrics on groups. Complete groups. Free topological groups*. (Russian) Uspehi Matem. Nauk (N.S.) 5, (1950). no. 2(36), 3-56.

[100] Griffeath D., Moore Ch. *New Constructions in Cellular automata,* Oxford University Press, 2003, 340 p. ISBN 9780195137187.

[101] Gunter C. *Universal profinite domains*, Inform. and Comput., vol. 72, nr. 1, 1987, p. 1–30.

[102] Gusfield D. *Algorithms on Strings, Trees and Sequences*. Computer Science and Computational Biology. New York: Cambridge University Press, 1997. 534 p. ISBN 0-521-58519-8.

[103] Gusfield D. *ReCombinatorics: The Algorithmics of Ancestral Recombination Graphs and Explicit Phylogenetic Networks*. London: The MIT Press, 2014. 580 p. ISBN 978-0-262-32447-2.

[104] Gusfield D., Irving R. W. *The Stable Marriage Problem: Structure and Algorithms*, Cambridge, MIT Press, 1989. 264 p. ISBN 9780262071185.

[105] Hall M. Jr. *A topology for free groups and related groups*, Ann. Math., vol.52, 1950, p. 127–139. ISSN 19398980.

[106] Hamming R. W. *Error Detecting and Error Correcting Codes*, Bell System Technical Journal, vol. 29, nr. 2, 1952, p. 147–160. ISSN 0005-8580.

[107] Hart J.B, Tsinakis C. *A concrete realization of the Hoare powerdomain*, Soft Comput., vol. 11, 2007, p. 1059-1063. ISSN 1432-7643.

[108] Hartman S., Mycielsky J. *On the embedding of topological groups into connected topological groups*. Colloq. Math., 1958, vol. 5, p. 167–169.

[109] Heath R. W. *On spaces with point countable bases*, Bull. Acad. Polon Sci., Ser Math., vol. 13, nr. 6 , 1965, p. 393–395.

[110] Heijne von G. *Sequence Analysis in Molecular Biology: Treasure Trove or Trivial Pursuit*. Academic Press, New York, 1987. 202 p. ISBN 978-0124337060.

[111] Hellerstein J.E., Koutsoupias E., Miranker D.P., Papadimitriou C.H., Samoladas V. *On a model of indexability and its bounds for range queries*, Journal of the ACM, vol. 49, 2002, nr. 1, p. 35–55. ISSN 1557-7351.

[112] Hellerstein J.E., Koutsoupias E., Miranker D.P., Papadimitriou C.H. *On the analysis of indexing schemes*, in: Proceedings of the Sixteenth ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems, Tucson, Arizona, 12-15 May 1997, Arizona, 1997, p. 249–256. ISBN 0-89791-910-6.

[113] Henriksen M., Raphael R., Woods R.G. *SP-scattered spaces; a new generalization of scattered spaces*, Comment. Math. Univ. Carolin., vol. 48, 2007, p. 487–505. ISSN 0010-2628.

[114] Herman G. T. *Geometry of Digital Spaces*, Birkhäuser, 1998. 203 p. ISBN 978-1-4612-4136-2.

[115] Higgins P. *Categories and groupoids*, Van Nostrand-Reinhold, New York, 1971.

[116] Hofmann K.H., Lawson J.D., Pym J.S. *The analytical and topological theory of semigroups*, de Gruyter, 1990. 400 p. ISBN 978-3-11-085604-0.

[117] Hoproft J. E., Ullman J. D. *Formal languages and their relation to automata*, Addison-Wesley, 1969. ISBN 978-0201029833.

[118] Hyers D. H. *Pseudonormed linear spaces and abelian groups*, Duke Math. Journal, vol. 5, 1939, p. 628–634. ISSN 0012-7094.

[119] James I., Shannon C.E. *30 April 1916 – 24 February 2001*. Biographical Memoirs of Fellows of the Royal Society. vol 55, p. 257–265. doi:10.1098/rsbm.2009.0015.

[120] Kakutani Sh. *Über die Metrisation der topologischen Gruppen*, Proc. Imp. Acad. Japan vol. 12, 1936, p. 82–84. ISSN 0021-4280.

[121] Kakutani Sh. *Free topological groups and infinite direct product topological groups*, Proceed. Imp. Acad. Tokyo vol. 20, 1944, p. 595–598. ISSN 0016-6601.

[122] Khalimsky E. *Topological structures in computer science*, Journal of Applied Math. and Simulation, vol. 1, 1987, nr. 1, p. 25–40.

[123] Khalimsky E., Kopperman R., Meyer P.R. *Computer graphics and connected topologies on finite ordered sets*, Topol. Appl., vol. 36, 1990, nr. 1, p. 1–17.

[124] Kolmogoroff A. *Zur Noimierbarkeit eines allgemeinen topologisehen linearen Räumes*, Studia mathematica, vol. 5, 1934, p. 29–33.

[125] Kotelnikov V. A. *Theory of Potential Noise Immunity*, 1956. 153 p.

[126] Kronheimer E.H. *The topology of digital images*, Topology and its Applications vol. 46, 1992, p. 279–303. ISSN 0166-8641.

[127] Lallement G., *Semigroups and combinatorial applications*, Wiley, 1979. ISBN 0471043796.

[128] Lamy J. *Integrating digital topology in image-processing libraries*, Computer Methods and Programs in Biomedicine, vol. 85, 2007, nr. 1, p. 51–58. ISSN 0169-2607.

[129] Larionova-Cojocaru I., Syrbu P. *On invariance of recursive differentiability under the isotopy of left Bol loops*, Proceedings of the 4-th Conference of Mathematical Society of the Republic of Moldova, dedicated to the centenary of V. Andrunachievici (1917-1997), Chisinau, June 28 – July 2, 2017, p. 117 – 120. ISBN 978-9975-71-915-5.

[130] Levenshtein V.I. *Binary codes capable of correcting deletions, insertions, and reversals*, DAN SSSR, vol. 163, nr. 4, p. 845–848, 1965 (in Russian) (English translation: *Soviet Physics – Doklady*, vol. 10, nr. 8, p. 707–710, 1966).

[131] Lind D., Marcus B. *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, 1995. ISBN 9780511626302.

[132] Macucci M. *Quantum Cellular automata: Theory, Experimentation and Prospects,* World Scientific, 2006. ISBN 978-1-908979-96-4.

[133] Maltsev A. I. *Free topological algebras*, Trans. Moscow Math. Soc., vol. 2, nr. 17, 1961, p. 173–200 (Russian original: Izvestia Akad. Nauk SSSR, vol. 21, 1957, p. 171–198).

[134] Maltsev A. I. *On the general theory of algebraic systems*, Mat. Sb. vol. 35, 1954, p. 3–20; English translation: Trans. Amer. Math. Soc., vol. 27, 1963, p. 125–148.

[135] Markov A. *On free topological groups*, Bulletin de l'Acadénie des Sciences de l'URSS vol. 9, 1945, p. 3–61 (In Russian) (English Translation: Topology and Topological Algebra, Transl. Amer. Math. Sot., Ser. I, vol. 8, 1962, p. 195–272.)

[136] Meinke K., Tuker J. V. *Universal Algebra*, in: S. Abramsky, D. Gabbay and T.S.E. Maibaum (eds), Handbook of Logic in Computer Science: Volume 1, Oxford University Press, 1993, p. 189–411. ISBN 9780198537618.

[137] Meinke K. *Universal algebra in higher types*, Theoretical Computer Science, vol. 100, 1992, p. 385–417. ISSN 0304-3975.

[138] Melin E. *Locally finite spaces and the join operator*, Proceedings of the 8th International Symposium on Mathematical Morphology, Rio de Janeiro, Brazil, Oct. 10-13, 2007, MCT/INPE, vol. 1, 2007, p. 63–74.

[139] Meseguer J., Montanari U., *Petri nets are monoids*, Information and Computation, vol. 88, 2000, p. 105–155. ISSN 0890-5401.

[140] Milnor J. *On the entropy geometry of cellular automata*, Complex Systems, vol. 2, nr. 3, 1988, p. 357–385.

[141] Minsky M. *Computation : Finite and infinite machines* Princeton, N.J.: Prentice Hall, 1967. ISBN:0-13-165563-9.

[142] Moore E. F. *Sequential Machines: Selected Papers*, Addison-Wesley, Reading, Mass., 1964.

[143] Montgomery Ph. *Logical Design of Digital Computers*, Wiley, 1958. ISBN 978-0471688051.

[144] Mostert P. S. *The structure of topological semigroups*. Revisited, Bull. Amer. Math. Soc. 72, nr. 4, 1966, p. 601–618. ISSN 0273-0979.

[145] Mukherjea A., Tserpes N. *Measures on topological semigroups*, Lect. notes in math. vol. 547, Springer, 1976, p. 67–172.

[146] Navarro G. *A guided tour to approximate string matching*, ACM Computing Surveys, vol. 33, no. 1, 2001. p. 31–88. ISSN 0360-0300.

[147] Nedev S.I. *o-metrizable spaces*, Trudy Moskov. Mat.Ob-va, vol. 24 1971, p. 201–236 (English translation: Trans. Moscow Math. Soc., vol 24, 1974, p. 213–247).

[148] Nedev S., Choban M. *On the theory of o-metrizable spaces, I*. Vestnik Moskovskogo Universiteta, 1972, nr. 1, p. 8–15. (English translation: Moscow University Mathematics Bulletin, vol. 27, 1973, nr. 1–2, p. 5–9).

[149] Nedev S., Choban M. *On the theory of o-metrizable spaces, II*. Vestnik Moskovskogo Universiteta, 1972, nr. 2, p. 10–17. (English translation: Moscow University Mathematics Bulletin, vol. 27, 1973, nr. 1–2, p. 65–70).

[150] Nedev S., Choban M. *On the theory of o-metrizable spaces, III*. Vestnik Moskovskogo Universiteta, 1972, nr. 3, p. 10–15. (English translation: Moscow University Mathematics Bulletin, vol. 27, 1973, nr. 3–4, p. 7–11).

[151] Needleman S. B., Wunsch C. D. *A general method applicable to the search for similarities in the amino acid sequence of two proteins*. Journal of Molecular Biology, vol. 48, nr. 3, 1970, p. 443–453. ISSN 0022-2836.

[152] Niemytzki, V. *On the third axiom of metric spaces*, Trans Amer. Math. Soc., vol. 29, 1927, p. 507–513.

[153] Niemytzki V. *Uber die Axiome des metriohen Raumes*, Math. Ann., vol. 104, 1931, p. 666–671.

[154] Olson M. V. *A time to sequence*. Science, 270:394-96, 1995.

[155] Paun Gh. *DNA Computing by Splicing and by Insertion-Deletion*, Handbook of Natural Computing, 2012, p. 1185–1202. ISBN 978-3-540-92909-3.

[156] Pestov V. *On the geometry of similarity search: dimensionality curse and concentration of measure*, Information Processing Letters, vol. 73, 2000, p. 47–51. ISSN: 0020-0190.

[157] Pilcher C. D., Wong J. K., Pillai S. K. *Inferring HIV transmission dynamics from phylogenetic sequence relationships*, PLoS Med., vol. 5, nr. 3, March 2008: e69, doi:10.1002/j.1538-7305.1950.tb00463.x.

[158] Pin J. E. *Polynomial closure of group languages and open sets of the Hall topology*, Theoret. Comput. Sci., vol. 169, 1996, p. 185–200. ISBN 978-3-540-58201-4.

[159] Pin J. E. *Finite group topology and p-adic topology for free monoids*, 12th ICALP, Lect. Notes Comp. Sci., vol. 194, 1985, p. 445–455. ISBN 3-540-15650-X.

[160] Pin J. E. *Topologies for the free monoid*, Journal of Algebra, vol. 137, 1991, p. 297–337. ISSN 0021-8693.

[161] Pinus A. G. *Hamiltonian closure on universal algebras*, Siberian Math. J., vol. 55, nr. 3, 2014, p. 498–502 . ISSN 0037-4466.

[162] Plotkin G. D. *A powerdomain construction*, SIAM Journal on Computing, vol. 5, 1976, p. 452–487. ISSN 0097-5397.

[163] Romaguera S., Sanchis M., Tkachenko M. *Free paratopological groups*, Topology Pro-ceed., vol. 27, 2003, nr. 2, p. 613–640.

[164] Roy N., Das A. *Topological Structure in Digital Image Processing : A Survey*, International Journal of Scientific Research in Computer Science, Engineering and Information Technology, vol. 2, 2017, nr. 6, p. 481–485. ISSN 2456-3307.

[165] Ruppert W.A.F. *Compact semitopological semigroups: an intrinsic theory*, Lect. notes in math. 1079, Springer, 1984. ISBN 978-3-540-38938-5.

[166] Sandhu R.S. *Lattice-based access control models*, Computer, vol. 26, nr. 11, 1993, p. 9–19. ISSN 0018-9162.

[167] Syrbu P., Ceban D. *On orthogonal systems of ternary quasigroups admitting nontrivial paratopies*, Quasigroups and Related Systems, nr. 1(25), 2017, p. 133–150. ISSN 1561-2848.

[168] Syrbu P., Izbash V. *Recursively differentiable quasigroups and complete recursive codes*, Commentat. Math. Univ. Carol. 45, nr. 2, 2004, p. 257–263.

[169] Syrbu P., Larionova-Cojocaru I. *On recursive differentiability of binary quasigroups*. Studia Universitatis Moldaviae. Revista Ştiinţifică a Universităţii de Stat din Moldova, 2015, nr.2(82), p. 53–60.

[170] Scott D. S. *Outline of a mathematical theory of computation*, Proc.4th Ann. Princeton Conference on Information Science and Systems, 1970, p. 169–176.

[171] Scott D. S. *The lattice of flow diagrams*, in: Symposium on Semantics of Algorithmic Languages, E. Engeler (Ed), Lecture Notes in Mathematics 188, Springer-Verlag, 1971, p. 311–366. ISBN 978-3-540-36499-3.

[172] Scott D. S. *Lattice theory, data types, and semantics, in Formal Semantics of Programming Languages*, R. Rustin (Ed.), Prentice Hall, Englewood Cliffs, New Jersey, 1972, p. 65–106.

[173] Scott D. S. *Continuous lattices*, in Toposes, algebraic geometry and logic, F. W. Lawvere (Ed), Lecture Notes in Mathematics, vol. 274, Springer-Verlag, 1972, p. 97–136. ISBN 978-3-540-37609-5.

[174] Scott D.S. *Data Types as lattices*, SIAM J. Computing 5, 1976, p. 522–587. ISSN 0097-5397.

[175] Scott D.S. *Domains for denotational semantics*, Lecture Notes in Computer Science, vol. 140, 1982, p. 577–613. ISBN 978-3-540-39308-5.

[176] Seguier J.-A. de. *Théorie des groupes finis. Élements de la Théorie des Groupes Abstraits*, Gauthier-Villars, Paris, 1904.

[177] Shannon C., *A Mathematical Theory of Communication*, The Bell System Technical Journal, vol. 27 (1948), p. 379-423, p. 623–656. https://doi.org/10.1002/j.1538-7305.1948.tb00917.x.

[178] Shcherbacov V. A. *Quasigroup based crypto-algorithms*, 2012, arXiv:1201.3016.

[179] Shcherbacov V. A. *Elements of Quasigroup Theory and Applications*. CRC Press, Boca Raton, 2017. ISBN 9781498721554.

[180] Shukla W., Srivastava A. K. *A Topology for Automata: A Note*, Information and Control, vol. 32, 1976, p. 163–168. ISSN: 0019-9958.

[181] Song Y. *An Introduction to Formal Languages and Machine Computation*, Singapore: World Scientific Publishing Co. Pte. Ltd., 1998. ISBN 9810221673.

[182] Smyth M. *The largest cartesian-closed category of domains*, Theoret. Comput. Sci., vol 27, 1983, p. 109–119. ISSN 0304-3975.

[183] Spreen D. *On Some Decision Problems in Programming*, Information and Computation, vol. 122, 1995, nr. 1, p. 120–139. ISSN 0890-5401.

[184] Swierczkowski S. *Topologies in free algebras*, Proceed. London Math. Soc., vol. 15, nr. 55, 1964, p. 566–576.

[185] Taylor W. *The clone of a topological space*, Heldman Verlag, 1986.

[186] Taylor W. *Incompatibility of algebra and topology*, Topology Atlas. Preprint, vol. 366, 1999, 64 p.

[187] Telgarsky R. *Total paracompactness and paracompact dispersed spaces* , Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys. vol. 16, 1968, p. 567–572.

[188] Telgarsky R. *C-scattered and paracompact spaces*, Fund. Math. vol. 73, 1971, p. 59–74.

[189] Telgarsky R. *On topological properties defined by games*, Preprint nr. 46, Institute Math. Polish Acad. Sci., 1972.

[190] Telgarsky R., *On topological properties defined by games*, Topics in Topology, Budapest, 1974, p. 617–624.

[191] Tychonoff A. *Über die topologinche Erweitenmgen von Ränmen*, Math. Ann., vol. 102, 1929, p. 544–561.

[192] **Vostokov S. V., Vostokova R. P., Budanaev I. A.** ***Asymmetric ID-Based Encryption System, Using an Explicit Pairing Function of the Reciprocity Law*****, Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica, vol. 2, 2016, p 40–44. ISSN 0236-3089.**

[193] **Vostokov S. V., Vostokova R. P., Budanaev I. A.** ***Asymmetric ID-Based Encryption System, using an Explicit Pairing Function of the Reciprocity*****, International Conference dedicated to the 70th anniversary of V. M. Levchuk, Algebra and Logic: Theory and Applications, Krasnoyarsk 2016, p. 16–18, 2016, ISBN 978–5–7638–3517–5.**

[194] Wallace A. D. *The structure of topological semigroups*, Bull. Amer. Math. Soc. vol. 61, 1955, nr. 2, p. 95–112. ISSN 1088-9485.

[195] Wang J. *Timed Petri Nets, Theory and Application*, Kluwer Academic Publishers, 1998. ISBN 978-1-4615-5537-7.

[196] Weickert J., *Efficient image segmentation using partial differential equations and morphology*, Pattern Recognition, vol. 34, 2001, p. 1813–1824. ISSN 0031-3203.

[197] Weil A. *Sur les espaces à structure uniiforme et sur la topologie générale*, Act. Sci. Ind., vol. 551, Paris, 1937.

[198] Whitehead A. N. *A Treatise on Universal Algebra*, 1898. ISBN 978-1429700320.

[199] Wilson W. A. *On Quasi-Metric Spaces*, American Journal of Mathematics, vol. 53, 1931, nr. 3, p. 675–84. ISSN 00029327.

[200] Worthington J. *Determinizing, Forgetting, and Automata in Monoidal Categories*, ASL North American Annual Meeting, March 17, 2010. ISSN 10798986.

DECLARATION OF LIABILITY

DECLARAȚIA PRIVIND ASUMAREA RĂSPUNDERII

Subsemnatul, Ivan Budanaev, declar pe răspundere personală că materialele prezentate în teza de doctorat "Distanţe pe Monoizi Liberi şi Aplicaţiile lor în Teoria Informaţiei" sunt rezultatul propriilor cercetări şi realizări ştiinţifice. Conştientizez că, în caz contrar, urmează să suport consecinţele în conformitate cu legislaţia în vigoare.

I hereby declare under my personal responsibility that the materials presented in Ph.D. thesis "Distances on Free Monoids and Their Applications in Theory of Information" are the result of my own research and scientific achievements. I realize that otherwise I am going to face the consequences according to the legislation in force.

Ivan Budanaev

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Publicații și teze la conferințe științifice

1. Choban M., Budanaev I., **"About the Construction of the Weighted Means of a Pair of Strings"**, Romai Journal, vol.14 n.1, 2018, pp. 73 – 87.

2. Budanaev I., **"About the Construction of the Bisector of Two Strings"**, Romai Journal, vol.13 n.2, 2017, pp. 1 – 11.

3. Choban M., Budanaev I., **"Efficiency and Penalty Factors on Monoids of Strings"**, Computer Science Journal of Moldova, vol.26 n.2 (77), 2018, pp. 99 – 114.

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5. Choban M., Budanaev I., **"Scattered and Digital Topologies in Image Processing"**, Proceedings of the Conference on Mathematical Foundations of Informatics, MFOI 2018, July 2-6, 2018, Chisinau, Republic of Moldova, Chisinau, pp. 21–40, 2018, ISBN: 978–9975–4237–7–9.

6. Budanaev I., **"About Non-Convexity of the Weighted Mean of a Pair of Strings"**, International Conference "Contemporary Trends in Science Development: Visions of Young Researchers", 7th Edition, Chisinau, 15 June 2018, Proceedings vol. 1, pp. 6–10.

7. Budanaev I., **"On the Midset of Pairs of Strings"**, International Conference on Mathematics, Informatics and Information Technologies Dedicated to the Illustrious Scientist Valentin Belousov, MITI 2018, Communications, pp. 134–135.

8. Choban M., Budanaev I., **"Measures of Similarity on Monoids of Strings"**, Conference on Mathematical Foundations of Informatics: Proceedings MFOI 2017, November 9-11, 2017, Chisinau, Republic of Moldova, Chisinau, Institute of Mathematics and Computer Science, pp. 51–58, 2017, ISBN: 978–9975–4237–6–82

9. Choban M.M., Budanaev I.A., **"Parallel Decompositions of Pairs of Strings and Their Applications"**, Presented in plenary at CAIM 2017: The 25^{th} Conference on Applied and Industrial Mathematics, Iaşi, Romania, September 14-17, 2017, Book of Abstracts, p. 64–65.

10. Budanaev I., **"On the Bisector of a Pair of Strings"**, Conference of Mathematical Society of Moldova: Proceedings CMSM4 2017, June 25 – July 2, 2017, Chisinău, Republic of Moldova, pp. 475–478, 2017, ISBN: 978-9975-71-915-5.

11. Budanaev I., **"Parallel Decompositions and The Weighted Mean of a Pair of Strings"**, International Conference "Contemporary Trends in Science Development: Visions of Young Researchers", 6th Edition, Chisinau, 15 June 2017, Proceedings pp. 7–11.

12. Budanaev I., **"On Hamming Type Distance Functions"**, CAIM 2016:The 24^{th} Conference on Applied and Industrial Mathematics, Craiova, Romania, September 15-18, 2016, Book of Abstracts, Editura SITECH, Craiova, pp. 15–16.

13. Budanaev I., **"On Hamming Type Distance Functions"**, Romai Journal v.12, no.2 (2016), pp. 25 – 32.

14. Choban M., Budanaev I., **"About Applications of Distances on Monoids of Strings"**, Computer Science Journal of Moldova, vol.24 n.3 (72), 2016, pp. 335 – 356.

15. Budanaev I., Choban M., **"Invariant Distances on Free Semigroups and Their Applications"**, *20th Annual Conference of the Mathematical Sciences Society of Romania*, Rezumatele Comunicărilor, A XX-a Conferinţă Anuală a Societăţii de Ştiinţe Matematice din România, Baia Mare, 19-22 mai 2016, pp. 17–18.

16. Choban M., Budanaev I., **"On the Theory of Free Topological Monoids and its Applications"**, International Conference "Mathematics & Information Technologies: Research and Education". MITRE 2016 Abstracts. Chisinau, June 23-26, 2016, pp. 21–22.

17. Vostokov S., Vostokova R., Budanaev I., **“Несимметричная система шифрования на идентификационных данных, использующая функцию явного спаривания закона взаимности”**, International Conference dedicated to the 70th anniversary of V. M. Levchuk, Algebra and Logic: Theory and Applications, Krasnoyarsk, Russian Federation, pp. 16–18, 2016, ISBN 978–5–7638–3517–5.

18. Choban M., Budanaev I., **“Distances on Monoids of Strings and Their Applications”**, Proceedings of the Conference on Mathematical Foundations of Informatics, MFOI 2016, July 25-29, 2016, Chisinau, Republic of Moldova, pp. 144–159, 2016, ISBN: 978–9975–4237–4–8.

19. Budanaev I., **“About Distance Functions of Hamming Type”**, International Conference *“Contemporary Trends in Science Development: Visions of Young Researchers”*, Conferinta Stiintifica a Doctoranzilor, IMI ASM, 2016, pp. 296 – 301.

20. Vostokov S., Vostokova R., Budanaev I., **“Asymmetric ID-Based Encryption System, Using an Explicit Pairing Function of the Reciprocity Law”**, Buletinul Academiei de Științe a Republicii Moldova. Matematica, n.2 (81), 2016, pp. 40 – 44.

Premii și distincţii

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